

Introduction

The exercises are taken from the book *How to prove it* by Daniel J Velleman, **2nd edition**, and the section, subsection and exercise numbers correspond exactly to the ones in the book.

Intended pace: This sheet contains more exercises than you are expected to complete in the time available. You are expected to try as many as you can, but you might have to skip some of them and complete them later on your own time.

Instead of doing all exercises in linear order, it is important that you try exercises from all chapters and sections. Here is a suggested pace for the exercises in this sheet:

- i) Wednesday afternoon: chapters 1 and 2. Start chapter 3 if possible
- ii) Thursday morning: chapter 3, start 4 if possible
- iii) Thursday afternoon: chapters 4, 5 and 6

1 Sentential Logic

1.1 Deductive Reasoning and Logical Connectives

Question 1— Analyze the logical forms of the following statements:

- a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.
- b) You won't go skiing, or you will and there won't be any snow.
- c) $\sqrt{7} \not\leq 2$.

Solution— In each case, we will identify the propositions and logical connectives involved, and then express the statements in symbolic form.

a) We can start by defining the following propositions:

- **R**: We'll have a reading assignment.
- **H**: We'll have homework problems.
- **T**: We'll have a test.

In the first sentence, we can focus on the following items:

We'll have **either** a reading assignment **or** homework problems

R
H

In this sentence, the words “either” and “or” indicate that we are dealing with a disjunction, which is represented by the logical connective $\vee$. Therefore, we can express this part of the

statement as:

$$R \vee H$$

The second part of the sentence states

we *won't* have **both** homework problems **and** a test

$\underbrace{\hspace{10em}}_H$ $\underbrace{\hspace{2em}}_T$

The words “both” and “and” indicate that we are dealing with a conjunction, which is represented by the logical connective $\wedge$. However, the sentence also includes the negation “we won’t have”, which indicates that we are dealing with a negation, represented by the logical connective $\neg$. Therefore, we can express this part of the statement as:

$$\neg(H \wedge T)$$

Now, these two sentences that we have analyzed are connected by the word “but”, which indicates that we are dealing with a conjunction, represented by the logical connective $\wedge$. Therefore, we can express the entire statement as:

$$(R \vee H) \wedge \neg(H \wedge T)$$

b) Using the same ideas as before, we can define the following propositions:

- S : You will go skiing.
- N : There will be snow.

The first part of the sentence, “You won’t go skiing”, can be expressed as $\neg S$. The second part, “you will and there won’t be any snow”, can be expressed as $S \wedge \neg N$. Both sentences are connected by the word “or”, so the entire statement can be expressed as:

$$\neg S \vee (S \wedge \neg N)$$

c) The idea is to use basic logical propositions $>$ and $=$ to express statements involving inequalities.

The statement $\sqrt{7} \not\leq 2$ can be expressed as $\neg(\sqrt{7} \leq 2)$. The statements $\sqrt{7} \leq 2$ can be expressed as $\sqrt{7} = 2$ or $\sqrt{7} < 2$. Thus, we obtain the final expression

$$\neg \left[(\sqrt{7} = 2) \vee (\sqrt{7} < 2) \right]$$

Question 2— Analyze the logical forms of the following statements

- a) Either John and Bill are both telling the truth, or neither of them is.
- b) I’ll have either fish or chicken, but I won’t have both fish and mashed potatoes.
- c) 3 is a common divisor of 6, 9, and 15.

Solution— For these exercises, we will proceed much faster than before, since we have already explained the reasoning behind the logical forms of the statements in the previous exercise.

- a) Let J : John is telling the truth, and B : Bill is telling the truth. The statement can be

expressed as:

$$(J \wedge B) \vee (\neg J \wedge \neg B)$$

- b) Let F : I'll have fish, C : I'll have chicken, and M : I'll have mashed potatoes. The statement can be expressed as:

$$(F \vee C) \wedge \neg(F \wedge M)$$

- c) We can use this exercise to introduce new notation that will be useful. Given two integers a and b , we say that a is a *divisor* of b if there exists an integer k such that $b = k \cdot a$. We denote this by $a \mid b$. With this notation, we can express the statement “3 is a common divisor of 6, 9, and 15” as:

$$3 \mid 6 \wedge 3 \mid 9 \wedge 3 \mid 15$$

Question 3— Analyze the logical forms of the following statements:

- a) Alice and Bob are not both in the room.
- b) Alice and Bob are both not in the room.
- c) Either Alice or Bob is not in the room.
- d) Neither Alice nor Bob is in the room.

Solution— Let A : Alice is in the room, and B : Bob is in the room.

- a) For each person, they can be either in the room or not in the room. The statement “Alice and Bob are not both in the room” is saying that it is not the case that both Alice and Bob are in the room simultaneously. This can be expressed as:

$$\neg(A \wedge B)$$

- b) The last statement was a negation: it said that there was a scenario, out of four possible ones, that was not true. By exchanging the words “not” and “both”, we can see that the statement “Alice and Bob are both not in the room” is now an affirmation: it is saying that both Alice and Bob are not in the room simultaneously. This can be expressed as:

$$(\neg A \wedge \neg B)$$

- c) The word “either” indicates that we are dealing with a disjunction, which is represented by the logical connective $\vee$. The statement “Either Alice or Bob is not in the room” is saying that at least one of Alice or Bob is not in the room. This can be expressed as:

$$\neg A \vee \neg B$$

- d) This last statement could be confused with the previous one because of the word “neither”. However, the statement “Neither Alice nor Bob is in the room” is saying that both Alice and Bob are not in the room, simultaneously. They are both outside the room. It is not a

disjunction, but a conjunction of two negations. This can be expressed as:

$$\neg A \wedge \neg B$$

Question 4— Which of the following expressions are well-formed formulas?

- a) $\neg(\neg P \vee \neg\neg R)$
- b) $\neg(P, Q, \wedge R)$
- c) $P \wedge \neg P$
- d) $(P \wedge Q)(P \vee R)$

Solution—

- a) This is a well-formed formula. Double negations are allowed, and the parentheses are correctly placed.
- b) This is not a well-formed formula. The comma is not a valid logical connective. Given this, the conjunction symbol is not well placed, as it is not clear what is the first operand.
- c) This is a well-formed formula. It represents a contradiction, but it is syntactically correct.
- d) This is not a well-formed formula. The juxtaposition of two formulas without any logical connective is not a valid logical expression. Note that in mathematics, we sometimes use juxtaposition to represent multiplication, but this is not well-defined in propositional logic.

Question 5— Let P stand for the statement “I will buy the pants” and S for the statement “I will buy the shirt” What English sentences are represented by the following expressions?

- a) $\neg(P \wedge \neg S)$
- b) $\neg P \wedge \neg S$
- c) $\neg P \vee \neg S$

Solution—

- a) The expression inside the negation can be read as “I will buy the pants and I will not buy the shirt”, or in a simpler way: “I will buy the pants without the shirt”. Thus, the negation of this can be formulated as: “I will not buy the pants without the shirt”.
- b) The expression $\neg P \wedge \neg S$ can be read as “I will not buy the pants and I will not buy the shirt”. This is a conjunction of two negations. To make it simpler, we can say: “I will buy neither the pants nor the shirt”.
- c) If we read the expression literally, we can say: “I will not buy the pants or I will not buy the shirt”. However, if you are shopping for clothes, you will most likely say something like “I won’t buy both the pants and the shirt”, which looks like an important transformation of the original expression but logically equivalent.

Question 7— Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Although you will have only your intuition

to guide you in answering this last question, in the next section we will develop some techniques for determining the validity of arguments.)

- a) Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.
- b) The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.
- c) Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.
- d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

Solution— In each item, we first cut the argument into propositions, then we identify the premises and conclusions and write it in a more “mathematical way”, and finally we analyze the logical forms of the arguments.

- a) *Jane and Pete won't both win the math prize. **Pete will win either the math prize or the chemistry prize.** Jane will win the math prize. Therefore, **Pete will win the chemistry prize.***

Seen in a more mathematical way, we can write the argument as follows:

Jane and Pete won't both win the math prize.
Pete will win either the math prize or the chemistry prize.
Jane will win the math prize.
Pete will win the chemistry prize.

From a validity perspective, the argument seems valid. If Jane is winning the math prize, then Pete cannot win it, and since he has to win either the math prize or the chemistry prize, he must win the chemistry prize.

- b) For this argument, we split it directly into premises and conclusion:

The main course will be either beef or fish.
The vegetable will be either peas or corn.
We will not have both fish as a main course and corn as a vegetable.
We will not have both beef as a main course and peas as a vegetable.

This argument looks invalid. The premises do not seem to imply the conclusion. For example, if we have beef and peas for dinner, then the premises are all satisfied: the main course is beef so the first premise is true, the vegetable is peas so the second premise is true, and it is also true that we did not have fish with corn. However, the conclusion is false: we did have both beef and peas. Therefore, the argument is invalid.

- c) Here is the argument in compact form:

Either John or Bill is telling the truth.
Either Sam or Bill is lying.
Either John is telling the truth or Sam is lying.

To see if this argument is valid, we can try to find a counterexample, that is, assign values to the different statements so that the premises are true but the conclusion is false.

For the conclusion to be false, we need John to be lying and Sam to be telling the truth.

With those assignments, if we need the first premise to be true, then Bill must be telling the truth.

However, this makes the second premise false, since Sam is telling the truth and Bill is also telling the truth.

On the other hand, if we want the second premise to be true given that John is lying and Sam is telling the truth, then Bill must be lying. But then the first premise is false, since both John and Bill are lying.

It seems that we cannot find a counterexample, so the argument seems to be valid.

d) Here is the argument in compact form:

Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy.
 Sales and expenses will not both go up.

Let us try to find a counterexample. For the conclusion to be false, we need both sales and expenses to go up.

If sales and expenses both go up and the boss is happy, then the first disjunct of the premise is true. Thus, the premise is true while the conclusion is false. This is a counterexample, so the argument is invalid.

1.2 Truth Tables

Question 2— Make truth tables for the following formulas:

a) $\neg P \vee Q$

b) $(S \vee G) \wedge (\neg S \vee \neg G)$

Solution— To simplify the reading of the truth tables, we will use the number 1 for true and 0 for false.

a) The truth table for $\neg P \vee Q$ is as follows:

P	Q	$\neg P$	$\neg P \vee Q$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

b) The truth table for $(S \vee G) \wedge (\neg S \vee \neg G)$ is as follows:

S	G	$S \vee G$	$\neg S$	$\neg G$	$(\neg S \vee \neg G)$	$(S \vee G) \wedge (\neg S \vee \neg G)$
1	1	1	0	0	0	0
1	0	1	0	1	1	1
0	1	1	1	0	1	1
0	0	0	1	1	1	0

Question 3— In this exercise we will use the symbol $+$ to mean exclusive or. In other words, $P + Q$ means “ P or Q , but not both.” Make a truth table for the formula $P + Q$, then find an equivalent formula that uses only the logical connectives $\neg$, $\wedge$ and $\vee$.

Solution— The truth table for $P + Q$ is as follows:

P	Q	$P + Q$
1	1	0
1	0	1
0	1	1
0	0	0

We can see that the formula $P + Q$ is true when P is true and Q is false, or when P is false and Q is true. This can be expressed as:

$$(P \wedge \neg Q) \vee (\neg P \wedge Q)$$

Question 4— Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.

Solution— We can use De Morgan’s law to find an equivalent formula using only the connectives $\wedge$ and $\neg$. According to De Morgan’s law, we have:

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Taking the negation of both sides, we get:

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

Now, let’s justify this with a truth table:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg(\neg P \wedge \neg Q)$
1	1	1	0	0	1
1	0	1	0	1	1
0	1	1	1	0	1
0	0	0	1	1	0

As we can see from the truth table, the columns for $P \vee Q$ and $\neg(\neg P \wedge \neg Q)$ are identical, confirming that they are equivalent.

Question 5— Some mathematicians use the symbol $\downarrow$ to mean nor. In other words, $P \downarrow Q$ means “neither P nor Q .” Make a truth table for the formula $P \downarrow Q$, then find an equivalent formula that uses only the logical connectives $\neg$, $\wedge$ and $\vee$.

Solution— The truth table for $P \downarrow Q$ is as follows:

P	Q	$P \downarrow Q$
1	1	0
1	0	0
0	1	0
0	0	1

We can see that the formula $P \downarrow Q$ is true only when both P and Q are false. This can be expressed as:

$$\neg P \wedge \neg Q$$

Question 6— Some mathematicians write $P|Q$ to mean “ P and Q are not both true.” (This connective is called *nand*, and is used in the study of circuits in computer science.) Make a truth table for the formula $P|Q$, then find an equivalent formula that uses only the logical connectives $\neg$, $\wedge$ and $\vee$.

Solution— The truth table for $P|Q$ is as follows:

P	Q	$P Q$
1	1	0
1	0	1
0	1	1
0	0	1

We can see that the formula $P|Q$ is true when at least one of P or Q is false. This can be expressed as:

$$\neg(P \wedge Q)$$

Question 7— Use truth tables to determine whether or not the following arguments are valid.

- Jane and Pete won’t both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.
- The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.
- Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling

the truth or Sam is lying.

- d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

Solution— For each argument, we will construct a truth table that includes the premises and the conclusion. If there is any row in which all the premises are true and the conclusion is false, then the argument is invalid. Otherwise, it is valid.

- a) Let J : Jane wins the math prize, P : Pete wins the math prize, and C : Pete wins the chemistry prize. The premises and conclusion can be expressed as:

- (i) Premise 1: $\neg(J \wedge P)$
- (ii) Premise 2: $P \vee C$
- (iii) Premise 3: J
- (iv) Conclusion: C

Now, we can construct the truth table:

J	P	C	$\neg(J \wedge P)$	$P \vee C$	J	C
1	1	1	0	1	1	1
1	1	0	0	1	1	0
1	0	1	1	1	1	1
1	0	0	1	0	1	0
0	1	1	1	1	0	1
0	1	0	1	1	0	0
0	0	1	1	1	0	1
0	0	0	1	0	0	0

In this truth table, we can see that there is no row in which all the premises are true and the conclusion is false. Therefore, the argument is valid.

- b) Let B : The main course is beef, F : The main course is fish, P : The vegetable is peas, and C : The vegetable is corn. The premises and conclusion can be expressed as:

- (i) Premise 1: $B \vee F$
- (ii) Premise 2: $P \vee C$
- (iii) Premise 3: $\neg(F \wedge C)$
- (iv) Conclusion: $\neg(B \wedge P)$

Now, we can construct the truth table:

B	F	P	C	$B \vee F$	$P \vee C$	$\neg(F \wedge C)$	$\neg(B \wedge P)$
1	1	1	1	1	1	0	0
1	1	1	0	1	1	1	0
1	1	0	1	1	1	0	1
1	1	0	0	1	0	1	1
1	0	1	1	1	1	0	0
1	0	1	0	1	1	1	0
1	0	0	1	1	1	0	1
1	0	0	0	1	0	1	1
0	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1
0	1	0	1	1	1	1	1
0	1	0	0	1	0	1	1
0	0	1	1	0	1	1	1
0	0	1	0	0	1	1	1
0	0	0	1	0	1	1	1
0	0	0	0	0	0	1	1

In this truth table, rows 2 and 6 have all the premises true and the conclusion false. Therefore, the argument is invalid. We can find a counterexample: if we have beef as the main course and peas as the vegetable, then all the premises are satisfied, but the conclusion is false. This counterexample can be obtained by looking at the problematic rows (in this case, row 6) and the values of the variables in those rows.

Note that row 2 is also problematic but is somehow less intuitive, as it suggests that we have both beef and fish as the main course. This is realistic or not depending on the situation that is being modeled.

- c) Let J : John is telling the truth, B : Bill is telling the truth, and S : Sam is telling the truth. The premises and conclusion can be expressed as:
- (i) Premise 1: $J \vee B$
 - (ii) Premise 2: $\neg S \vee \neg B$
 - (iii) Conclusion: $J \vee \neg S$

Now, we can construct the truth table:

J	B	S	$J \vee B$	$\neg S \vee \neg B$	$J \vee \neg S$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	1	0
0	0	0	0	1	1

In this truth table, we can see that there is no row in which all the premises are true and the conclusion is false. Therefore, the argument is valid.

d) Let S : Sales will go up, E : Expenses will go up, and H : The boss will be happy. The premises and conclusion can be expressed as:

(i) Premise 1: $(S \wedge H) \vee (E \wedge \neg H)$

(ii) Conclusion: $\neg(S \wedge E)$

Now, we can construct the truth table:

S	E	H	$(S \wedge H) \vee (E \wedge \neg H)$	$\neg(S \wedge E)$
1	1	1	1	0
1	1	0	1	0
1	0	1	1	1
1	0	0	0	1
0	1	1	0	1
0	1	0	1	1
0	0	1	0	1
0	0	0	0	1

The first two rows have all the premises true and the conclusion false. Therefore, the argument is invalid. We can find a counterexample: if sales and expenses both go up, and the boss is happy, then the premise is satisfied, but the conclusion is false. This counterexample can be obtained by looking at the problematic rows (in this case, row 1) and the values of the variables in those rows.

Question 11— Use the laws stated in the text to find simpler formulas equivalent to these formulas.

a) $\neg(\neg P \wedge \neg Q)$

b) $(P \wedge Q) \vee (P \wedge \neg Q)$

c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution— In each case, we will apply one rule at a time, and we will justify each step with the name of the law that was applied.

a)

$$\begin{aligned}\neg(\neg P \wedge \neg Q) &\equiv \neg\neg P \vee \neg\neg Q && \text{(De Morgan's law)} \\ &\equiv P \vee Q && \text{(Double negation)}\end{aligned}$$

b)

$$\begin{aligned}(P \wedge Q) \vee (P \wedge \neg Q) &\equiv P \wedge (Q \vee \neg Q) && \text{(Distributive law)} \\ &\equiv P \wedge T && \text{(Negation law)} \\ &\equiv P && \text{(Identity law)}\end{aligned}$$

c)

$$\begin{aligned}\neg(P \wedge \neg Q) \vee (\neg P \wedge Q) &\equiv (\neg P \vee Q) \vee (\neg P \wedge Q) && \text{(De Morgan's law)} \\ &\equiv (\neg P \vee Q) && \text{(Absorption law)}\end{aligned}$$

Question 16— Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	?
1	1	1
1	0	1
0	1	0
0	0	1

Solution— The formula $P \vee \neg Q$ has the required truth values:

P	Q	$\neg Q$	$P \vee \neg Q$
1	1	0	1
1	0	1	1
0	1	0	0
0	0	1	1

Thus $P \vee \neg Q$ is a formula with the desired truth table.

Question 18— Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

Solution— If the conclusion of an argument is a tautology, then the argument is valid no matter the premises. Indeed, for the argument to be invalid, there must be a scenario in which all the premises are true and the conclusion is false. However, if the conclusion is a tautology, it is always true, and therefore there cannot be a scenario in which it is false. Thus, the argument is valid always. This kind of argument is called a *trivial argument*.

If the conclusion of an argument is a contradiction, then the argument is invalid no matter the

premises, if there are premises that can be true.

If one of the premises is a tautology, nothing particular happens: the argument can be valid or invalid, depending on the other premises and the conclusion.

On the other hand, if one of the premises is a contradiction, then the argument is valid no matter the other premises or the conclusion, because there cannot be a scenario in which all the premises are true, and therefore the argument is valid by default. This kind of argument is called a *vacuous argument*.

1.3 Variables and Sets

Question 5— Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

- a) $-3 \in \{x \in \mathbb{R} | 13 - 2x > 1\}$.
- b) $4 \in \{x \in \mathbb{R}^- | 13 - 2x > 1\}$.
- c) $5 \notin \{x \in \mathbb{R} | 13 - 2x > c\}$.

Solution— a) The statement involves the property $P(x) := 13 - 2x > 1$. The universe of discourse is the set of real numbers. The truth set of the property $P(x)$ is exactly $\{x \in \mathbb{R} | 13 - 2x > 1\}$, and in this latter statement, x is a bound variable. The statement we are analyzing is saying that -3 is in this truth set.

To verify if -3 is in the truth set, we can check if $P(-3)$ is true:

$$P(-3) := 13 - 2(-3) > 1 \equiv 13 + 6 > 1 \equiv 19 > 1$$

This statement is true. There are no free variables in this statement.

- b) This statement is very similar to the previous one, except that now we are looking at the truth set of the property $P(x)$ restricted to the *negative* real numbers. The statement is saying that 4 is in this truth set. However, 4 is not a negative real number, and therefore it cannot be in the truth set. There are no free variables in this statement, and it is false.

Notice that even if it is true that $13 - 2(4) > 1$, the falseness of the statement comes from the fact that 4 is not in the universe of discourse for the statement, which is $\mathbb{R}^-$.

- c) In this statement, there is a new variable c that is free, and the property $P(x) := 13 - 2x > c$ is true or false depending on the value of c . The statement is saying that 5 is not in the truth set of this property. However, we cannot say whether this statement is true or false without knowing the value of c .

Question 6— Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

- a) $w \in \{x \in \mathbb{R} | 13 - 2x > c\}$.
- b) $4 \in \{x \in \mathbb{R} | 13 - 2x \in \{y | y \text{ is a prime number}\}\}$.
- c) $4 \in \{x \in \{y | y \text{ is a prime number}\} | 13 - 2x > 1\}$.

Solution— It is important in this exercise to simplify the statements, as they are written in a very convoluted way.

- a) The statement is asking if w belongs to a certain set. Not only w is a free variable, but also c in the definition of the set. The variable x is bound to the statement. We cannot determine if the statement is true or false without knowing the values of w and c .
- b) We start from the inside of the statement. The set $\{y|y \text{ is a prime number}\}$ is the set of all prime numbers, which we can denote P for simplicity. This confirms that y is a bound variable. The statement is asking if 4 belongs to the set of real numbers x such that $13 - 2x$ is a prime number. We can check if $13 - 2(4)$ is a prime number: $13 - 8 = 5$, which is prime. Therefore, the statement is true. The variable x is bound to the statement, and there are no free variables.
- c) Again, we start from the inside of the statement. The set $\{y|y \text{ is a prime number}\}$ is the set of all prime numbers, which we can denote P for simplicity. This confirms that y is a bound variable.

The statement $x \in \{y|y \text{ is a prime number}\}$ is equivalent to “ x is prime” or $x \in P$. We can then rewrite

$$\{x \in \{y|y \text{ is a prime number}\} | 13 - 2x > 1\} = \{x \in P | 13 - 2x > 1\}$$

In other words, the statement is asking if 4 is a prime number and if $13 - 2(4) > 1$. The first part of the statement is false, as 4 is not a prime number. Therefore, the whole statement is false. The variable x is bound to the statement, and there are no free variables.

Question 8— What are the truth sets of the following statements? List a few elements of the truth set if you can.

- a) x is a real number and $x^2 - 4x + 3 = 0$.
- b) x is a real number and $x^2 - 2x + 3 = 0$.
- c) x is a real number and $5 \in \{y \in \mathbb{R} | x^2 + y^2 < 50\}$.

Solution— a) We are looking for a subset of the real numbers, whose elements satisfy the equation $x^2 - 4x + 3 = 0$. We can factor the left-hand side of the equation to find the solutions:

$$x^2 - 4x + 3 = (x - 1)(x - 3) = 0$$

This means that $x = 1$ or $x = 3$. Therefore, the truth set is $\{1, 3\}$.

- b) We are looking for a subset of the real numbers, whose elements satisfy the equation $x^2 - 2x + 3 = 0$. We can use the quadratic formula:

$$x = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2}$$

Since the discriminant is negative, there are no real solutions. Therefore, the truth set is the empty set $\emptyset$.

- c) We are looking for the real numbers x such that 5 is in the set $\{y \in \mathbb{R} | x^2 + y^2 < 50\}$. This

means that we need to check which values of x would satisfy $x^2 + 5^2 < 50$, which simplifies to $x^2 + 25 < 50$, or $x^2 < 25$. Therefore, the truth set is the interval $(-5, 5)$.

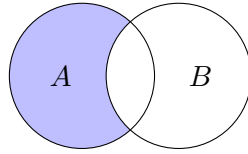
1.4 Operations on Sets

Question 4— Use Venn diagrams to verify the following identities:

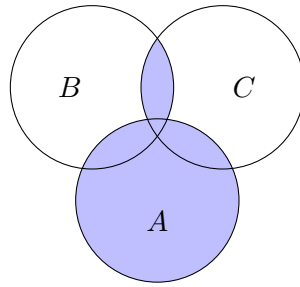
- a) $A \setminus (A \cap B) = A \setminus B$
b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution— In each diagram, the shaded portion is the portion produced by both sides of the corresponding identity.

$$A \setminus (A \cap B) = A \setminus B$$



$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



Question 5— Verify the identities in the previous exercise by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

Solution—

a)

$$\begin{aligned}
 x \in A \setminus (A \cap B) &\equiv x \in A \wedge x \notin (A \cap B) && \text{(Definition of set difference)} \\
 &\equiv x \in A \wedge \neg(x \in A \wedge x \in B) && \text{(Definition of intersection)} \\
 &\equiv x \in A \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's law)} \\
 &\equiv (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) && \text{(Distributive law)} \\
 &\equiv F \vee (x \in A \wedge x \notin B) \\
 &\equiv x \in A \wedge x \notin B && \text{(Identity law)} \\
 &\equiv x \in A \setminus B && \text{(Definition of set difference)}
 \end{aligned}$$

b)

$$\begin{aligned}
 x \in A \cup (B \cap C) &\equiv x \in A \vee x \in (B \cap C) && \text{(Definition of union)} \\
 &\equiv x \in A \vee (x \in B \wedge x \in C) && \text{(Definition of intersection)} \\
 &\equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) && \text{(Distributive law)} \\
 &\equiv x \in (A \cup B) \cap (A \cup C) && \text{(Definition of union and intersection)}
 \end{aligned}$$

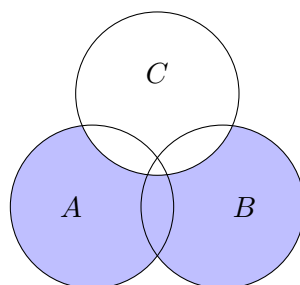
Question 6— Use Venn diagrams to verify the following identities:

a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

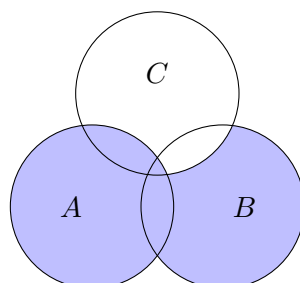
b) $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$

Solution— In each diagram, the shaded portion is the portion produced by both sides of the corresponding identity.

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$$



$$A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$$



Question 7— Verify the identities in the previous exercise by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

Solution—

a)

$$\begin{aligned}
x \in (A \cup B) \setminus C &\equiv x \in (A \cup B) \wedge x \notin C && \text{(Definition of set difference)} \\
&\equiv (x \in A \vee x \in B) \wedge x \notin C && \text{(Definition of union)} \\
&\equiv (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C) && \text{(Distributive law)} \\
&\equiv x \in (A \setminus C) \vee x \in (B \setminus C) && \text{(Definition of set difference)} \\
&\equiv x \in (A \setminus C) \cup (B \setminus C) && \text{(Definition of union)}
\end{aligned}$$

b)

$$\begin{aligned}
x \in A \cup (B \setminus C) &\equiv x \in A \vee x \in (B \setminus C) && \text{(Definition of union)} \\
&\equiv x \in A \vee (x \in B \wedge x \notin C) && \text{(Definition of set difference)} \\
&\equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \notin C) && \text{(Distributive law)} \\
&\equiv x \in (A \cup B) \wedge \neg(x \notin A \wedge x \in C) && \text{(De Morgan's law)} \\
&\equiv x \in (A \cup B) \wedge (x \notin (C \setminus A)) && \text{(Definition of set difference)} \\
&\equiv x \in (A \cup B) \setminus (C \setminus A) && \text{(Definition of set difference)}
\end{aligned}$$

1.5 The Conditional and Biconditional

Question 1— Analyze the logical forms of the following statements:

- If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.
- Having both a fever and a headache is a sufficient condition for George to go to the doctor.
- Both having a fever and having a headache are sufficient conditions for George to go to the doctor.
- If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.

Solution— In each case, we will identify the propositions and logical connectives involved, and then express the statements in symbolic form. Recall that the expression “ P is a sufficient condition for Q ” can be expressed as $P \rightarrow Q$, which can also be stated as “ Q is a necessary condition for P ”.

- Let S : This gas has an unpleasant smell, E : This gas is explosive, and H : This gas is hydrogen. The statement can be expressed as:

$$(S \vee \neg E) \rightarrow \neg H$$

- Let F : George has a fever, H : George has a headache, and D : George goes to the doctor. The statement can be expressed as:

$$(F \wedge H) \rightarrow D$$

Note that this proposition states that it is the combination of having a fever and having a headache that is a sufficient condition for George to go to the doctor. In other words, if George has only one of these symptoms, he may not go to the doctor.

- Using the same propositions as before, the statement can be expressed as:

$$(F \rightarrow D) \wedge (H \rightarrow D)$$

Notice that in this case, we have two sufficient conditions for George to go to the doctor: having a fever and having a headache. Each, individually, is enough to make George go to the doctor.

- d) Let P : x is prime, and O : x is odd. The statement can be expressed as:

$$(x \neq 2) \rightarrow (P \rightarrow O)$$

Note that this statement is true, and could be simplified as “If $x \neq 2$ and x is prime, then x is odd”. This suggests that the following statement is logically equivalent to the original one:

$$(x \neq 2) \wedge P \rightarrow O$$

Question 2— Analyze the logical forms of the following statements:

- Mary will sell her house only if she can get a good price and find a nice apartment.
- Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.
- John will kill himself, unless someone stops him. (Hint: First try to rephrase this using the words if and then instead of unless.)
- If x is divisible by either 4 or 6, then it isn't prime.

Solution— a) Let S : Mary will sell her house, P : Mary can get a good price, and A : Mary can find a nice apartment. The statement can be expressed as:

$$S \rightarrow (P \wedge A)$$

One might be tempted to write the statement as $(P \wedge A) \rightarrow S$, but this is not correct.

We cannot conclude that Mary will sell the house if she can get a good price and find a nice apartment, as there may be other reasons why she would sell or other factors that are not mentioned. What we can conclude, is that in the case in which Mary sells her house, we can be sure that she was able to get a good price and find a nice apartment.

- b) Let C : Having a good credit history, D : Having an adequate down payment, and M : Getting a mortgage. The statement can be expressed as:

$$M \rightarrow (C \wedge D)$$

Again, we must be careful not to confuse this statement with its reciprocal. Even if one has good credit score and enough money for the down payment, we could fail to get the mortgage. However, if we have the evidence that the mortgage was given, then we can be sure that the person had good credit score and enough money for the down payment.

- c) Let K : John kills himself and S : someone stops John. The statement can be rewritten as “if

nobody stops John, then he kills himself”:

$$\neg S \rightarrow K.$$

d) Let D_4 : x is divisible by 4, D_6 : x is divisible by 6, and P : x is prime. The statement is:

$$(D_4 \vee D_6) \rightarrow \neg P.$$

Question 5—

- a) Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
- b) Show that $(P \rightarrow Q) \vee (P \rightarrow R)$ is equivalent to $P \rightarrow (Q \vee R)$.

Solution— We will use the fact that $P \rightarrow Q$ is equivalent to $\neg P \vee Q$.

a)

$$\begin{aligned} P \leftrightarrow Q &\equiv (P \rightarrow Q) \wedge (Q \rightarrow P) && \text{(Definition of biconditional)} \\ &\equiv (\neg P \vee Q) \wedge (\neg Q \vee P) && \text{(Implication equivalence)} \\ &\equiv [(\neg P \vee Q) \wedge \neg Q] \vee [(\neg P \vee Q) \wedge P] && \text{(Distributive law)} \\ &\equiv (P \wedge Q) \vee (\neg P \wedge \neg Q) && \text{(Distributive and Contradiction laws)} \end{aligned}$$

b)

$$\begin{aligned} (P \rightarrow Q) \vee (P \rightarrow R) &\equiv (\neg P \vee Q) \vee (\neg P \vee R) && \text{(Implication equivalence)} \\ &\equiv \neg P \vee (Q \vee R) && \text{(Associative and commutative laws)} \\ &\equiv P \rightarrow (Q \vee R) && \text{(Implication equivalence)} \end{aligned}$$

2 Quantificational Logic

2.1 Quantifiers

Question 1— Analyze the logical forms of the following statements.

- a) Anyone who has forgiven at least one person is a saint.
- b) Nobody in the calculus class is smarter than everybody in the discrete math class.
- c) Everyone likes Mary, except Mary herself.
- d) Jane saw a police officer, and Roger saw one too.
- e) Jane saw a police officer, and Roger saw him too.

Solution—

- a) Let $F(x)$: x has forgiven at least one person, and $S(x)$: x is a saint. The statement can be expressed as:

$$\forall x(F(x) \rightarrow S(x))$$

- b) Let $C(x)$: x is in the calculus class, $D(x)$: x is in the discrete math class, and $S(x, y)$: x is smarter than y . The statement can be expressed as:

$$\neg \exists x (C(x) \wedge \forall y (D(y) \rightarrow S(x, y)))$$

- c) Let $M(x)$: x likes Mary. The statement can be expressed as:

$$\forall x (x \neq \text{Mary} \rightarrow M(x)) \wedge \neg M(\text{Mary})$$

- d) Let $P(x)$: x is a police officer, and $S(x, y)$: x saw y . The statement can be expressed as:

$$\exists x (P(x) \wedge S(\text{Jane}, x)) \wedge \exists y (P(y) \wedge S(\text{Roger}, y))$$

- e) Using the same propositions as before, the statement can be expressed as:

$$\exists x (P(x) \wedge S(\text{Jane}, x) \wedge S(\text{Roger}, x))$$

Notice the difference with the previous statement: in this case, we are saying that Jane and Roger saw the same police officer, while in the previous statement, they could have seen the same one or different police officers.

Question 3— Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

- Every number that is larger than x is larger than y .
- For every number a , the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq -2$.
- All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.
- If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10 .

Solution—

- a) This can be expressed as:

$$\forall z (z > x \rightarrow z > y)$$

The free variables are x and y .

We could simplify the statement by taking out the bound variable z and writing the statement as $x \geq y$.

- b) We could express this as:

$$\forall a (\exists z (az^2 + 4z - 2 = 0) \leftrightarrow a \geq -2)$$

- c) We could express this as:

$$\forall z (z^3 - 3z < 3 \rightarrow z < 10)$$

There are no free variables.

d) We could express this as:

$$(\exists x(x^2 + 5x = w) \wedge \exists y(4 - y^2 = w)) \rightarrow -10 < w < 10$$

The variable w is free; x and y are bound.

Question 4— Translate the following statements into idiomatic English.

- a) $\forall x[(H(x) \wedge \neg \exists y M(x, y)) \rightarrow U(x)]$, where $H(x)$ means “ x is a man”, $M(x, y)$ means “ x is married to y ”, and $U(x)$ means “ x is unhappy”.
- b) $\exists z(P(z, x) \wedge S(z, y) \wedge W(y))$, where $P(z, x)$ means “ z is a parent of x ”, $S(z, y)$ means “ z and y are siblings”, and $W(y)$ means “ y is a woman”.

Solution— a) The statement can be expressed as: “For every man x , if x is not married to anyone, then x is unhappy”. In other words, every single man is unhappy.
b) The statement can be expressed as: “There exists a person z such that z is a parent of x and z and y are siblings and y is a woman”. Note that this statement has two free variables, x and y , and therefore it is not a statement, but rather a predicate. We can rephrase the statement as: “There exists a person z such that z is a parent of x and z has a sister y ”.

Question 6— Are these statements true or false? The universe of discourse is the set of all people, and $P(x, y)$ means “ x is a parent of y .”

- a) $\exists x \forall y P(x, y)$.
- b) $\forall x \exists y P(x, y)$.
- c) $\neg \exists x \exists y P(x, y)$.
- d) $\exists x \neg \exists y P(x, y)$.
- e) $\exists x \exists y \neg P(x, y)$.

Solution— a) False. There is no person who is a parent of everyone.
b) False. Not every person is the parent of someone.
c) False. There are people who are parents of someone.
d) True. There are people who are not parents of anyone.
e) True. We can find two people such that one of them is not the parent of the other.

Question 7— Are these statements true or false? The universe of discourse is $\mathbb{N}$.

- a) $\forall x \exists y (2x - y = 0)$.
- b) $\exists y \forall x (2x - y = 0)$.
- c) $\forall x \exists y (x - 2y = 0)$.
- d) $\forall x (x < 10 \rightarrow \forall y (y < x \rightarrow y < 9))$.

e) $\exists y \exists z (y + z = 100)$.

f) $\forall x \exists y (y > x \wedge \exists z (y + z = 100))$.

- Solution—**
- a) True. For every natural number x , we can find a natural number y such that $2x - y = 0$. In fact, we can take $y = 2x$ which is a natural number.
 - b) False. There is no single natural number y such that $2x - y = 0$ for all natural numbers x .
 - c) False. Not for every natural number x , we can find a natural number y such that $x - 2y = 0$. In fact, we can take $y = \frac{x}{2}$ which is a natural number only when x is even. On the other hand, such y will not exist for odd x .
 - d) True. If $x < 10$, then the largest value of x is 9. Thus, for any $y < x$, we have $y < 9$. Note that this would not be true if the universe of discourse was $\mathbb{R}$ or $\mathbb{Q}$.
 - e) True. We can take $y = 50$ and $z = 50$, so that $y + z = 100$. There are actually many more possible values for y and z that satisfy this equation, such as $y = 0$ and $z = 100$, or $y = 1$ and $z = 99$, etc.
 - f) False. Take $x = 101$. Then it is impossible to find a natural number y such that $y > 101$ and $y + z = 100$ for some natural number z . In fact, if $y > 101$, then z would have to be negative, which is not a natural number.

Question 8— Repeat the previous exercise, but this time the universe of discourse is $\mathbb{R}$. Then change the universe of discourse to $\mathbb{Z}$ and repeat the exercise again.

Solution— For the universe $\mathbb{R}$, the six answers are respectively: true, false, true, false, true, true. For (1), choose $y = 2x$; for (3), choose $y = x/2$; for (5), choose $y = z = 50$; and for (6), choose any $y > x$ and then $z = 100 - y$. The second statement is false because one fixed y cannot equal $2x$ for every x , and the fourth is false for $x = 9.5, y = 9.4$.

For the universe $\mathbb{Z}$, the six answers are true, false, false, true, true, true. In (1) choose $y = 2x$, and in (5) choose $y = z = 50$. Statement (3) fails for odd x ; for (6), choose $y = x + 1$ and $z = 99 - x$.

2.2 Equivalences Involving Quantifiers

Question 1— Negate these statements and then reexpress the results as equivalent positive statements.

- a) Everyone who is majoring in math has a friend who needs help with his homework.
- b) Everyone has a roommate who dislikes everyone.
- c) $A \cup B \subseteq C \setminus D$.
- d) $\exists x \forall y (y > x \rightarrow \exists z (z^2 + 5z = y))$.

- Solution—**
- a) The negation of the statement is: “There exists a person who is majoring in math and does not have a friend who needs help with his homework”. We can reexpress this as: “There exists a math major who has no friends that need help with their homework”.
 - b) The negation of the statement is: “There exists a person who does not have a roommate who dislikes everyone”.

- c) The negation of the statement is: “There exists an element in $A \cup B$ that is not in $C \setminus D$ ”. We can reexpress this as: “There exists an element in $A \cup B$ that is either not in C or is in D ”.
- d) The negation of the statement is: “For every real number x , there exists a real number y such that $y > x$ and for every real number z , $z^2 + 5z \neq y$ ”. We can reexpress this as: “For every real number x , there exists a real number y greater than x such that the equation $z^2 + 5z = y$ has no solution for z ”.

Question 3— Are these statements true or false? The universe of discourse is $\mathbb{N}$.

- a) $\forall x(x < 7 \rightarrow \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.
- b) $\exists! x((x - 4)^2 = 9)$.
- c) $\exists! x((x - 4)^2 = 25)$.
- d) $\exists x \exists y((x - 4)^2 = 25 \wedge (y - 4)^2 = 25)$.

Solution— a) True. For every natural number x less than 7, we can find natural numbers a , b , and c such that $a^2 + b^2 + c^2 = x$.

$$\begin{aligned}
0 &= 0^2 + 0^2 + 0^2 \\
1 &= 1^2 + 0^2 + 0^2 \\
2 &= 1^2 + 1^2 + 0^2 \\
3 &= 1^2 + 1^2 + 1^2 \\
4 &= 2^2 + 0^2 + 0^2 \\
5 &= 2^2 + 1^2 + 0^2 \\
6 &= 2^2 + 1^2 + 1^2
\end{aligned}$$

- b) False. The equation $(x - 4)^2 = 9$ has two solutions: $x - 4 = 3$ or $x - 4 = -3$. Therefore, there are two values of x that satisfy the equation: $x = 7$ and $x = 1$.
- c) True. The equation $(x - 4)^2 = 25$ has two solutions: $x - 4 = 5$ or $x - 4 = -5$. Therefore, there is only one value of x that satisfies the equation in the universe of discourse $\mathbb{N}$: $x = 9$.
- d) True. The only value in $\mathbb{N}$ that satisfies this equation is $x = 9$. Therefore, we can take $x = 9$ and $y = 9$ to satisfy the statement. Notice that the statement does not require x and y to be different.

Question 5— Show that $\neg \exists x \in A, P(x)$ is equivalent to $\forall x \in A, \neg P(x)$.

Solution—

$$\begin{aligned}
\neg \exists x \in A, P(x) &\equiv \neg \exists x(x \in A \wedge P(x)) && \text{(Definition of } \in) \\
&\equiv \forall x \neg(x \in A \wedge P(x)) && \text{(Negation of } \exists) \\
&\equiv \forall x(x \notin A \vee \neg P(x)) && \text{(De Morgan's law)} \\
&\equiv \forall x(x \in A \rightarrow \neg P(x)) && \text{(Implication equivalence)} \\
&\equiv \forall x \in A, \neg P(x) && \text{(Definition of } \in)
\end{aligned}$$

Question 6— Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x(P(x) \vee Q(x))$ is equivalent to $\exists xP(x) \vee \exists xQ(x)$. [Hint: In this section of the book, the distribution of the universal quantifier over conjunction is discussed. You can use that result to prove this one.]

Solution— We are going to negate the statement two times. For space considerations, we will not use the symbol $\equiv$ in each line, as we have been doing before. Instead, we will just write the logical equivalences in each line and the reason for the equivalence on the right side of the line.

$$\begin{array}{ll}
 \exists x(P(x) \vee Q(x)) & \text{(Given)} \\
 \neg\neg\exists x(P(x) \vee Q(x)) & \text{(Double negation)} \\
 \neg\forall x\neg(P(x) \vee Q(x)) & \text{(Negation of } \exists \text{)} \\
 \neg\forall x(\neg P(x) \wedge \neg Q(x)) & \text{(De Morgan's law)} \\
 \neg(\forall x\neg P(x) \wedge \forall x\neg Q(x)) & \text{(Distribution of } \forall \text{ over conjunction)} \\
 \neg(\forall x\neg P(x)) \vee \neg(\forall x\neg Q(x)) & \text{(De Morgan's law)} \\
 (\exists xP(x)) \vee (\exists xQ(x)) & \text{(Negation of } \forall \text{)}
 \end{array}$$

Question 7— Show that $\exists x(P(x) \rightarrow Q(x))$ is equivalent to $\forall xP(x) \rightarrow \exists xQ(x)$.

Solution— For space considerations, we will not use the symbol $\equiv$ in each line, as we have been doing before. Instead, we will just write the logical equivalences in each line and the reason for the equivalence on the right side of the line.

$$\begin{array}{ll}
 \exists x(P(x) \rightarrow Q(x)) & \text{(Given)} \\
 \exists x(\neg P(x) \vee Q(x)) & \text{(Implication equivalence)} \\
 \exists x(\neg P(x)) \vee \exists x(Q(x)) & \text{(Distribution of } \exists \text{ over disjunction)} \\
 (\neg\forall xP(x)) \vee (\exists xQ(x)) & \text{(Negation of } \forall \text{)} \\
 (\forall xP(x)) \rightarrow (\exists xQ(x)) & \text{(Implication equivalence)}
 \end{array}$$

Question 8— Show that $(\forall x \in A, P(x)) \wedge (\forall x \in B, P(x))$ is equivalent to $\forall x \in (A \cup B), P(x)$.

Solution— For space considerations, we will not use the symbol $\equiv$ in each line, as we have been doing before. Instead, we will just write the logical equivalences in each line and the reason for the equivalence on the right side of the line.

$$\begin{array}{ll}
 (\forall x \in A, P(x)) \wedge (\forall x \in B, P(x)) & \text{(Given)} \\
 (\forall x(x \in A \rightarrow P(x))) \wedge (\forall x(x \in B \rightarrow P(x))) & \text{(Definition of } \in \text{)} \\
 \forall x((x \in A \rightarrow P(x)) \wedge (x \in B \rightarrow P(x))) & \text{(Distribution of } \forall \text{ over conjunction)} \\
 \forall x((\neg(x \in A) \vee P(x)) \wedge (\neg(x \in B) \vee P(x))) & \text{(Implication equivalence)} \\
 \forall x((\neg(x \in A) \wedge (\neg(x \in B))) \vee P(x)) & \text{(Distributive law)} \\
 \forall x(\neg(x \in A \vee x \in B) \vee P(x)) & \text{(De Morgan's law)} \\
 \forall x((x \in A \vee x \in B) \rightarrow P(x)) & \text{(Implication equivalence)} \\
 \forall x \in (A \cup B), P(x) & \text{(Definition of } \in \text{)}
 \end{array}$$

Question 9— Is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall xP(x) \vee \forall xQ(x)$? Explain. (Hint: Try assigning meanings to $P(x)$ and $Q(x)$.)

Solution— No, they are not equivalent.

For example, let the universe of discourse be the set of all people, and define two mutually exclusive predicates: $P(x)$: x has no siblings, and $Q(x)$: x has at least one sibling. Then, the statement $\forall x(P(x) \vee Q(x))$ is true, since every person either has no siblings or has at least one sibling. However, the statement $\forall xP(x) \vee \forall xQ(x)$ is false, since there are some people who have no siblings and some people who have at least one sibling. Not all people can be put into either category. Therefore, the two statements are not equivalent. In other words, the universal quantifier does not distribute over disjunction.

Question 10— Show that the statements $A \subseteq B$ and $A \setminus B = \emptyset$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

Solution— On one hand, we can rewrite the first statement as follows:

$$\begin{aligned} A \subseteq B &\equiv \forall x(x \in A \rightarrow x \in B) && \text{(Definition of subset)} \\ &\equiv \forall x(\neg(x \in A) \vee x \in B) && \text{(Implication equivalence)} \end{aligned}$$

The second statement can be rewritten as follows:

$$\begin{aligned} A \setminus B = \emptyset &\equiv \forall x(x \notin A \setminus B) && \text{(Definition of empty set)} \\ &\equiv \forall x(\neg(x \in A) \vee x \in B) && \text{(Definition of set difference + negation)} \end{aligned}$$

Since both statements can be rewritten as $\forall x(\neg(x \in A) \vee x \in B)$, we conclude that they are equivalent.

3 Proofs

3.1 Proof Strategies

Question 1— Consider the following theorem.

Theorem: Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.

- Identify the hypotheses and conclusion of the theorem. Are the hypotheses true when $n = 6$? What does the theorem tell you in this instance? Is it right?
- What can you conclude from the theorem in the case $n = 15$? Check directly that this conclusion is correct.
- What can you conclude from the theorem in the case $n = 11$?

Solution—

- The hypotheses of the theorem are: n is an integer larger than 1 and n is not prime. The conclusion of the theorem is: $2^n - 1$ is not prime.

When $n = 6$, the hypotheses are true because 6 is an integer larger than 1 and 6 is not prime. The theorem tells us that, given that both premises are true, we can conclude that $2^6 - 1 = 63$ is not prime. It is indeed true that 63 is not prime, since $63 = 7 \cdot 9$.

- b) When $n = 15$, the hypotheses are true because 15 is an integer larger than 1 and 15 is not prime. The theorem tells us that, given that both premises are true, we can conclude that $2^{15} - 1 = 32767$ is not prime. It is indeed true that 32767 is not prime, since $32767 = 7 \cdot 31 \cdot 151$.
- c) In the case $n = 11$, the hypotheses are false because 11 is prime. Therefore, we cannot conclude anything about the primality of $2^{11} - 1 = 2047$ from this theorem.

Question 2— Consider the following theorem. (The theorem is correct, but we will not ask you to prove it here.)

Theorem: Suppose that $a \neq 0$ and $b^2 > 4ac$. Then the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.

- a) Identify the hypotheses and conclusion of the theorem.
- b) To give an instance of the theorem, you must specify values for a , b , and c , but not x . Why?
- c) What can you conclude from the theorem in the case $a = 2$, $b = -5$, $c = 3$? Check directly that this conclusion is correct.
- d) What can you conclude from the theorem in the case $a = 2$, $b = 4$, $c = 3$?

- Solution**— a) The hypotheses of the theorem are: $a \neq 0$ and $b^2 > 4ac$. The conclusion of the theorem is: the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions, in other words, that there exist exactly two different values of x that satisfy the equation.
- b) In terms of free or bound variables, a , b , and c are free variables, while x is a bound variable. x is simply the variable that we are trying to solve for, and it is not part of the hypotheses of the theorem. Therefore, we cannot specify a value for x in order to give an instance of the theorem.
 - c) In the case $a = 2$, $b = -5$, and $c = 3$, we can check that the hypotheses are true:

$$b^2 = (-5)^2 = 25 > 4 \cdot 2 \cdot 3 = 24 = 4ac$$

Therefore, the theorem applies, and we can conclude that the quadratic equation $2x^2 - 5x + 3 = 0$ has exactly two real solutions.

We can check this directly by solving the equation:

$$2x^2 - 5x + 3 = 0 \implies x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{5 \pm \sqrt{25 - 24}}{4} = \frac{5 \pm 1}{4}$$

Therefore, the two solutions are $x = \frac{5+1}{4} = \frac{6}{4} = \frac{3}{2}$ and $x = \frac{5-1}{4} = \frac{4}{4} = 1$. We have found exactly two real solutions, as predicted by the theorem.

- d) In the case $a = 2$, $b = 4$, and $c = 3$, we can check that the hypotheses are false:

$$b^2 = 4^2 = 16 < 4 \cdot 2 \cdot 3 = 24 = 4ac$$

Therefore, the theorem does not apply, and we cannot conclude anything about the number of real solutions of the quadratic equation $2x^2 + 4x + 3 = 0$ from this theorem. In fact, we can check directly that this equation has no real solutions:

$$2x^2 + 4x + 3 = 0 \implies x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{-4 \pm \sqrt{16 - 24}}{4} = \frac{-4 \pm \sqrt{-8}}{4}$$

Since the discriminant is negative, there are no real solutions to this equation.

Note that we cannot conclude that there are no real solutions from the theorem, since the hypotheses of the theorem are false in this case. It could be the case that the equation has one real solution, or two real solutions, or no real solutions. The theorem only tells us what happens when the hypotheses are true.

Question 3— Consider the following incorrect theorem:

Incorrect Theorem: Suppose n is a natural number larger than 2, and n is not a prime number. Then $2n + 13$ is not a prime number.

What are the hypotheses and conclusion of this theorem? Show that the theorem is incorrect by finding a counterexample.

Solution— The hypotheses of the theorem are: n is a natural number larger than 2, and n is not a prime number. The conclusion of the theorem is: $2n + 13$ is not a prime number.

To show that the theorem is incorrect, we need to find a counterexample. We can take $n = 4$, which is a natural number larger than 2 and is not prime. Then, we can check the conclusion:

$$2n + 13 = 2 \cdot 4 + 13 = 8 + 13 = 21$$

Since 21 is not a prime number, this does not serve as a counterexample.

Let's try $n = 9$, which is also a natural number larger than 2 and is not prime. Then, we can check the conclusion:

$$2n + 13 = 2 \cdot 9 + 13 = 18 + 13 = 31$$

Since 31 is a prime number, this serves as a counterexample to the theorem in which the hypotheses are true but the conclusion is false. Therefore, the theorem is incorrect.

Question 4— Complete the following alternative proof of the theorem:

Theorem: Suppose a and b are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$.

Proof (incomplete): Suppose $0 < a < b$. Then $b - a > 0$. [Fill in a proof of $b^2 - a^2 > 0$ here.] Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore if $0 < a < b$ then $a^2 < b^2$.

Remark

To start doing these very simple proofs, it is worth remembering what are the things we assume are true for real numbers. Here is a small list of things we can use in our proofs:

- (i) If $a < b$ and $b < c$, then $a < c$.
- (ii) If $a < b$ and $c > 0$, then $ac < bc$.
- (iii) If $a < b$ and $c < 0$, then $ac > bc$.
- (iv) If $a < b$, then $a + c < b + c$.
- (v) If $a > 0$ and $b > 0$, then $ab > 0$.
- (vi) $a < b$ if and only if $a - b < 0$.

Solution— We will write the whole proof for readability:

Suppose $0 < a < b$. Then $b - a > 0$. Note that it is also true that $b + a > 0$ since both b and a are positive.

Now, we can factor $b^2 - a^2$ as $(b - a)(b + a)$, which according to the previous two statements is the product of two positive numbers. Therefore, $b^2 - a^2 > 0$.

Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore if $0 < a < b$ then $a^2 < b^2$.

Question 5— Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

Solution— Suppose $a < b < 0$. Then $b - a > 0$ since $b > a$. Note that it is also true that $b + a < 0$ since both b and a are negative.

Now, we can factor $b^2 - a^2$ as $(b - a)(b + a)$, which according to the previous two statements is the product of a positive number and a negative number. Therefore, $b^2 - a^2 < 0$, which implies that $a^2 - b^2 > 0$, or equivalently, that $a^2 > b^2$.

Therefore if $a < b < 0$ then $a^2 > b^2$.

Alternative proof: Suppose $a < b < 0$. Then $0 < -b < -a$ since multiplying both sides of an inequality by a negative number reverses the inequality. Since $-b$ and $-a$ are both positive, we can apply the previous theorem to conclude that $(-b)^2 < (-a)^2$, or equivalently, that $b^2 < a^2$. Therefore if $a < b < 0$ then $a^2 > b^2$.

Question 6— Suppose a and b are real numbers. Prove that if $0 < a < b$ then $\frac{1}{b} < \frac{1}{a}$.

Solution— Suppose $0 < a < b$.

Notice that if $a > 0$, then $\frac{1}{a} > 0$. This is true because $a \frac{1}{a} = 1 > 0$, so supposing that $\frac{1}{a} \leq 0$ would lead to a contradiction.

Similarly, if $b > 0$, then $\frac{1}{b} > 0$. Therefore, both $\frac{1}{a}$ and $\frac{1}{b}$ are positive.

Now, we can multiply both sides of the inequality $0 < a < b$ by $\frac{1}{ab}$, which is positive since both a and b are positive. This gives us:

$$0 < \frac{1}{b} < \frac{1}{a}$$

Alternative proof: First transform the goal into something simpler:

$$\frac{1}{b} < \frac{1}{a} \iff 0 < \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$$

Thus, we want to show that $\frac{b-a}{ab} > 0$. Since $0 < a < b$, we have that $b-a > 0$ and $ab > 0$. Therefore, $\frac{b-a}{ab} > 0$, which implies that $\frac{1}{b} < \frac{1}{a}$.

Question 8— Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$, then $x \in B$.

Solution— Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$.

We want to prove that if $x \notin D$, then $x \in B$, which is a goal of the form “if P , then Q ”. To prove such a goal, our most basic strategy is to assume that P is true and then try to prove that Q is true. Another way is to prove the contrapositive, which is logically equivalent to the original statement. The contrapositive of “if P , then Q ” is “if not Q , then not P ”. We will use the contrapositive strategy here.

We will assume that $x \notin B$ and try to prove that $x \in D$.

We have the following information: $A \setminus B \subseteq C \cap D$, $x \in A$, and $x \notin B$.

As $x \in A$ and $x \notin B$, we can conclude that $x \in A \setminus B$. Since $A \setminus B \subseteq C \cap D$, we can conclude that $x \in C \cap D$. Therefore, $x \in D$.

Notice that trying a direct proof seems much more complicated.

Question 9— Suppose a and b are real numbers. Prove that if $a < b$ then $\frac{a+b}{2} < b$.

Solution— We want to prove that $\frac{a+b}{2} < b$. Our goal can be manipulated to obtain something that is easier to prove:

$$\frac{a+b}{2} < b \iff a+b < 2b \iff a < b$$

The rightmost statement is exactly our assumption. Therefore, the only thing we need to do is to write a proper proof doing the reverse steps:

Suppose $a < b$. Then $a+b < b+b = 2b$. Dividing both sides of the inequality by $2 > 0$ gives us $\frac{a+b}{2} < \frac{2b}{2} = b$. Therefore, if $a < b$ then $\frac{a+b}{2} < b$.

Question 11— Suppose a, b, c , and d are real numbers, $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Solution— Our goal is of the form $P \rightarrow Q$. One way to see that the contrapositive is interesting is to notice that $\neg Q$ gives us $c \leq d$, which is a simpler statement than $ac \geq bd$. Therefore, we will prove the contrapositive of our goal.

Suppose $a, b, c, d \in \mathbb{R}$, with $0 < a < b$, $d > 0$, and suppose that $c \leq d$. We want to prove that $ac < bd$.

We start from $0 < a < b$. Now we need to consider two cases: $c \leq 0$ and $c > 0$.

Case 1: $c \leq 0$. Then $ac \leq 0$ and $bd > 0$, so $ac < bd$.

Case 2: $c > 0$. Then we can multiply both sides of the inequality by c to get $0 < ac < bc$. Since $c \leq d$, we can also multiply both sides of the inequality by b to get $bc \leq bd$. Combining these two inequalities gives us $0 < ac < bc \leq bd$, which implies that $ac < bd$.

Question 12— Suppose x and y are real numbers, and $3x + 2y \leq 5$. Prove that if $x > 1$ then $y < 1$.

Solution— Suppose x and y are real numbers, and $3x + 2y \leq 5$. Let us suppose that $x > 1$, and try to prove that $y < 1$.

Multiply the inequality $x > 1$ by 3 to get $3x > 3$. Then add $2y$ to both sides of the inequality to get $3x + 2y > 3 + 2y$. Since we also have that $3x + 2y \leq 5$, we can combine these two inequalities to get $3 + 2y < 3x + 2y \leq 5$.

From here, we can conclude that $3 + 2y < 5$, which implies that $2y < 2$, and therefore $y < 1$.

Question 16— Consider the following incorrect theorem:

Incorrect Theorem: Suppose that x and y are real numbers and $x \neq 3$. If $x^2y = 9y$ then $y = 0$.

a) What's wrong with the following proof of the theorem?

Proof. Suppose that $x^2y = 9y$. Then $(x^2 - 9)y = 0$. Since $x \neq 3$, $x^2 \neq 9$, so $x^2 - 9 \neq 0$. Therefore we can divide both sides of the equation $(x^2 - 9)y = 0$ by $x^2 - 9$, which leads to the conclusion that $y = 0$. Thus, if $x^2y = 9y$ then $y = 0$.

□

b) Show that the theorem is incorrect by finding a counterexample.

Solution— a) The proof makes an incorrect step when it claims that “Since $x \neq 3$, $x^2 \neq 9$, so $x^2 - 9 \neq 0$ ”. This is not true, since x could be -3 , which is not equal to 3, but $(-3)^2 - 9 = 0$. The statement $x \neq 3 \rightarrow x^2 \neq 9$ is false.

b) A counterexample is $x = -3$ and $y = 1$. Then $x \neq 3$, $x^2y = 9 \cdot 1 = 9$, and $9y = 9 \cdot 1 = 9$, so the condition is satisfied. However, $y = 1 \neq 0$, which shows that the conclusion is not necessarily true.

3.2 Proofs Involving Negations and Conditionals

Question 1— This problem could be solved by using truth tables, but don't do it that way. Instead, use the methods for writing proofs discussed so far in this chapter.

- Suppose $P \rightarrow Q$ and $Q \rightarrow R$ are both true. Prove that $P \rightarrow R$ is true.
- Suppose $\neg R \rightarrow (P \rightarrow \neg Q)$ is true. Prove that $P \rightarrow (Q \rightarrow R)$ is true.
- Suppose $P \rightarrow Q$ and $R \rightarrow \neg Q$ are both true. Prove that $P \rightarrow \neg R$ is true.
- Suppose that P is true. Prove that $Q \rightarrow \neg(Q \rightarrow \neg P)$ is true.

Solution—

- a) Suppose $P \rightarrow Q$ and $Q \rightarrow R$ are both true. We want to prove that $P \rightarrow R$ is true. To do this, we will assume that P is true and try to prove that R is true. Since P is true and $P \rightarrow Q$ is true, we can conclude that Q is true. Since Q is true and $Q \rightarrow R$ is true, we can conclude that R is true. Therefore, if P is true then R is true, which proves that $P \rightarrow R$ is true.
- b) We will prove this by contradiction. Suppose $\neg R \rightarrow (P \rightarrow \neg Q)$ is true, and suppose that $P \rightarrow (Q \rightarrow R)$ is false. Then, by the definition of implication, it is false when P is true and $Q \rightarrow R$ is false. Since $Q \rightarrow R$ is false, we can conclude that Q is true and R is false. Thus, at this point, we have that P is true, Q is true, and R is false. Since we assumed that $\neg R \rightarrow (P \rightarrow \neg Q)$ is true, and we have that R is false, we can conclude that $P \rightarrow \neg Q$ is true. However, we also have that P is true and thus we can conclude that $\neg Q$ is true. This is a contradiction, since we also have that Q is true. The contradiction arose from our assumption that $P \rightarrow (Q \rightarrow R)$ is false. Therefore, we can conclude that $P \rightarrow (Q \rightarrow R)$ is true.
- c) We will again prove this by contradiction. Suppose $P \rightarrow Q$ and $R \rightarrow \neg Q$ are both true, and suppose that $P \rightarrow \neg R$ is false. Then, by the definition of implication, it is false when P is true and $\neg R$ is false. Since $\neg R$ is false, we can conclude that R is true. Thus, at this point, we have that P is true and R is true. Using the implications we have as givens, we have that P is true and $P \rightarrow Q$ is true, so we can conclude that Q is true. We also have that R is true and $R \rightarrow \neg Q$ is true, so we can conclude that $\neg Q$ is true. This is a contradiction, since we also have that Q is true. The contradiction arose from our assumption that $P \rightarrow \neg R$ is false. Therefore, we can conclude that $P \rightarrow \neg R$ is true.
- d) In this question, our assumptions are very simple, but our goal is complicated. We will try to simplify our goal by using the definition of implication. We want to prove that $Q \rightarrow \neg(Q \rightarrow \neg P)$ is true. By the definition of implication, this is equivalent to proving that $\neg Q \vee \neg(Q \rightarrow \neg P)$ is true. Using the definition of implication again, we can rewrite this as $\neg Q \vee \neg(\neg Q \vee \neg P)$. Using De Morgan's law, we can rewrite this as $\neg Q \vee (Q \wedge P)$. Finally, using the distributive law, we can rewrite this as $(\neg Q \vee Q) \wedge (\neg Q \vee P)$. Since $(\neg Q \vee Q)$ is a tautology, we can conclude that $(\neg Q \vee Q) \wedge (\neg Q \vee P)$ is equivalent to $\neg Q \vee P$. Since we assumed that P is true, we can conclude that $\neg Q \vee P$ is true. Therefore, we have shown that $Q \rightarrow \neg(Q \rightarrow \neg P)$ is true.

Question 3— Suppose $A \subseteq C$, and B and C are disjoint. Prove that if $x \in A$ then $x \notin B$.

Solution— Suppose $A \subseteq C$, and B and C are disjoint. We want to prove that if $x \in A$ then $x \notin B$. To do this, we will assume that $x \in A$ and try to prove that $x \notin B$. Since $A \subseteq C$, we can conclude that $x \in C$. Since B and C are disjoint, we can conclude that $x \notin B$. Therefore, if $x \in A$ then $x \notin B$.

Question 4— Suppose that $A \setminus B$ is disjoint from C and $x \in A$. Prove that if $x \in C$ then $x \in B$.

Solution— Suppose that $A \setminus B$ is disjoint from C and $x \in A$. We want to prove that if $x \in C$ then $x \in B$. We will use the contrapositive strategy.

Suppose that $x \notin B$. Since $x \in A$ and $x \notin B$, we can conclude that $x \in A \setminus B$. Since $A \setminus B$ is disjoint from C , we can conclude that $x \notin C$. Therefore, if $x \notin B$ then $x \notin C$, which is the contrapositive of our goal. Thus, if $x \in C$ then $x \in B$.

Question 5— Prove the following result by contradiction:

Suppose $A \cap C \subseteq B$ and $a \in C$. Then $a \notin A \setminus B$.

Solution— Suppose that $A \cap C \subseteq B$ and $a \in C$. Now suppose by contradiction that $a \in A \setminus B$. Then, by the definition of set difference, we have that $a \in A$ and $a \notin B$.

Given that $a \in A$ and $a \in C$, we can conclude that $a \in A \cap C$. Since $A \cap C \subseteq B$, we can conclude that $a \in B$. This is a contradiction, since we also have that $a \notin B$.

Question 11— Consider the following incorrect theorem:

Incorrect Theorem: Suppose x and y are real numbers and $x + y = 10$. Then $x \neq 3$ and $y \neq 8$.

a) What's wrong with the following proof of the theorem?

Proof. Suppose the conclusion of the theorem is false. Then $x = 3$ and $y = 8$. But then $x + y = 11$, which contradicts the given information that $x + y = 10$. Therefore the conclusion must be true.

□

b) Show that the theorem is incorrect by finding a counterexample.

Solution— a) The proof makes a mistake when negating the conclusion. The negation of " $x \neq 3$ and $y \neq 8$ " is " $x = 3$ or $y = 8$ ", not " $x = 3$ and $y = 8$ ". Therefore, the proof is incorrect.

b) A counterexample is $x = 3$ and $y = 7$. Then $x + y = 3 + 7 = 10$, which satisfies the hypothesis of the theorem. However, $x = 3$, which contradicts the conclusion of the theorem. Therefore, the theorem is incorrect.

Question 12— Consider the following incorrect theorem:

Incorrect Theorem: Suppose that $A \subseteq C$, $B \subseteq C$, and $x \in A$. Then $x \in B$.

a) What's wrong with the following proof of the theorem?

Proof. Suppose that $x \notin B$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \notin B$ and $B \subseteq C$, $x \notin C$. But now we have proven both $x \in C$ and $x \notin C$, so we have reached a contradiction. Therefore $x \in B$.

□

b) Show that the theorem is incorrect by finding a counterexample.

Solution— a) The proof is incorrect because it uses the two givens $x \notin B$ and $B \subseteq C$ to conclude that $x \in C$. This is not a valid inference. The fact that $x \notin B$ does not imply that $x \notin C$, even if $B \subseteq C$. Therefore, the proof is incorrect. More generally, from $P \rightarrow Q$ and $\neg P$ we cannot conclude $\neg Q$.

b) A counterexample is $A = \{1\}, B = \{2\}, C = \{1, 2\}$, and $x = 1$. Then $x \in A$ and $A \subseteq C$, but $x \notin B$. Therefore, the theorem is incorrect.

Question 13— Use truth tables to show that *modus tollens* is a valid rule of inference.

Solution— Modus tollens is the rule of inference that states that if $P \rightarrow Q$ is true and $\neg Q$ is true, then $\neg P$ is true. We can show that this is a valid rule of inference by constructing a truth table for $P, Q, P \rightarrow Q, \neg Q$, and $\neg P$.

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$
1	1	1	0	0
1	0	0	1	0
0	1	1	0	1
0	0	1	1	1

From the truth table, we can see that whenever $P \rightarrow Q$ is true and $\neg Q$ is true, then $\neg P$ is also true. Therefore, modus tollens is a valid rule of inference.

3.3 Proofs Involving Quantifiers

Question 2— Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

Solution— Let $x \in A \cap B$. If $x \notin C$, then $x \in B \setminus C$, contradicting that A and $B \setminus C$ are disjoint. Hence $x \in C$, so $A \cap B \subseteq C$.

Question 3— Prove that if $A \subseteq B \setminus C$, then A and C are disjoint.

Solution— If $x \in A \cap C$, then $x \in A \subseteq B \setminus C$, so $x \notin C$, a contradiction. Thus $A \cap C = \emptyset$.

Question 4— For any set A , the set $\mathcal{P}(A)$ is the set of all subsets of A .

Suppose $A \subseteq \mathcal{P}(A)$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

Solution— Let $X \in \mathcal{P}(A)$. Then $X \subseteq A$. Since $A \subseteq \mathcal{P}(A)$, every element of X belongs to $\mathcal{P}(A)$. Hence $X \subseteq \mathcal{P}(A)$, so $X \in \mathcal{P}(\mathcal{P}(A))$.

Question 7— Prove that for every real number x , if $x > 2$, then there is a real number y such that $y + \frac{1}{y} = x$.

Solution— Let $x > 2$ and choose $y = (x + \sqrt{x^2 - 4})/2$. Then $y > 0$ and $y^2 - xy + 1 = 0$. Dividing by y gives $y + 1/y = x$.

Question 8— A family of sets is a set whose elements are sets. For example, $\mathcal{F} = \{\{1, 2, 3\}, \{3\}, \{3, 4, 5, 6\}\}$ is a family of sets.

The *union* of a family of sets $\mathcal{F}$ is the set $\cup\mathcal{F} = \{x : x \in A \text{ for some } A \in \mathcal{F}\}$. For example, if $\mathcal{F} = \{\{1, 2, 3\}, \{3\}, \{3, 4, 5, 6\}\}$, then

$$\cup\mathcal{F} = \{1, 2, 3, 4, 5, 6\}.$$

The *intersection* of a family of sets $\mathcal{F}$ is the set $\cap\mathcal{F} = \{x : x \in A \text{ for all } A \in \mathcal{F}\}$. For example, if $\mathcal{F} = \{\{1, 2, 3\}, \{3\}, \{3, 4, 5, 6\}\}$, then

$$\cap\mathcal{F} = \{3\}.$$

For more details, see Section 2.3, *More Operations on Sets*, in the textbook.

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.

Solution— If $x \in A$, then $A \in \mathcal{F}$ witnesses the definition of the union, so $x \in \cup\mathcal{F}$. Therefore $A \subseteq \cup\mathcal{F}$.

Question 9— Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cap\mathcal{F} \subseteq A$.

Solution— If $x \in \cap\mathcal{F}$, then x belongs to every member of $\mathcal{F}$. Since $A \in \mathcal{F}$, we get $x \in A$. Thus $\cap\mathcal{F} \subseteq A$.

Question 10— Suppose that $\mathcal{F}$ is a nonempty family of sets, B is a set, and $\forall A \in \mathcal{F} (B \subseteq A)$. Prove that $B \subseteq \cap\mathcal{F}$.

Solution— Let $x \in B$. For every $A \in \mathcal{F}$, the hypothesis gives $x \in A$. Hence x belongs to every member of $\mathcal{F}$, so $x \in \cap\mathcal{F}$.

Question 11— Suppose that $\mathcal{F}$ is a family of sets. Prove that if $\emptyset \in \mathcal{F}$, then $\cap\mathcal{F} = \emptyset$.

Solution— If $x \in \cap\mathcal{F}$, then in particular $x \in \emptyset$, because $\emptyset \in \mathcal{F}$. This is impossible, so $\cap\mathcal{F} = \emptyset$.

Question 12— Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$, then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

Solution— Let $x \in \cup\mathcal{F}$. Then $x \in A$ for some $A \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{G}$, we have $A \in \mathcal{G}$, and hence $x \in \cup\mathcal{G}$.

Question 13— Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$, then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Solution— Let $x \in \cap \mathcal{G}$. Then x belongs to every member of $\mathcal{G}$, and therefore to every member of the subfamily $\mathcal{F}$. Thus $x \in \cap \mathcal{F}$.

Question 18— In this problem all variables range over $\mathbb{Z}$, the set of all integers.

- a) Prove that if $a|b$ and $a|c$, then $a|(b+c)$.
- b) Prove that if $ac|bc$ and $c \neq 0$, then $a|b$.

Solution— a) Write $b = ak$ and $c = al$ for integers k, l . Then $b + c = a(k + l)$, so $a | (b + c)$.
b) Since $ac | bc$, there is an integer m with $bc = mac$. As $c \neq 0$, cancellation gives $b = ma$, so $a | b$.

Question 19—

- a) Prove that for all real numbers x and y there is a real number z such that $x + z = y - z$.
- b) Would the statement in part (a) be correct if “real number” were changed to “integer”? Justify your answer.

Solution— a) Choose $z = (y - x)/2$. Then $x + z = (x + y)/2 = y - z$.
b) No. For $x = 0$ and $y = 1$, the equation requires $z = 1/2$, which is not an integer.

Question 20— Consider the following theorem:

Theorem: For every real number x , $x^2 \geq 0$.

What’s wrong with the following proof of the theorem?

Proof: Suppose not. Then for every real number x , $x^2 < 0$. In particular, plugging in $x = 3$ we would get $9 < 0$, which is clearly false. This contradiction shows that for every number x , $x^2 \geq 0$.

□

Solution— The proof makes a first mistake when negating the statement. The negation of “for every real number x , $x^2 \geq 0$ ” is “there exists a real number x such that $x^2 < 0$ ”, not “for every real number x , $x^2 < 0$ ”. Therefore, the proof is incorrect. Indeed, a correct proof would negate the statement and get to “there exists a real number x such that $x^2 < 0$ ”.

Question 21— Consider the following incorrect theorem:

Theorem: If $\forall x \in A, x \neq 0$ and $A \subseteq B$, then $\forall x \in B, x \neq 0$.

- a) What’s wrong with the following proof of the theorem?

Proof: Let x be an arbitrary element of A . Since $\forall x \in A, x \neq 0$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, we have $x \in B$. Since $x \in B$ and $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B, x \neq 0$.

□

- b) Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Solution— a) The proof is incorrect because it assumes that an arbitrary element of A is also an arbitrary element of B . This is not true, since there may be elements of B that are not in A . Therefore, the proof is incorrect.

- b) A counterexample is $A = \{1\}$ and $B = \{0, 1\}$. Then $\forall x \in A, x \neq 0$ and $A \subseteq B$, but $\forall x \in B, x \neq 0$ is false since $0 \in B$. Therefore, the theorem is incorrect.

Question 22— Consider the following incorrect theorem:

Theorem: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy^2 = y - x$.

What's wrong with the following proof of the theorem?

Proof: Let $x = \frac{y}{y^2+1}$. Note that this is well defined because $y^2 + 1 > 0$ for all $y \in \mathbb{R}$. Then

$$y - x = y - \frac{y}{y^2 + 1} = \frac{y(y^2 + 1) - y}{y^2 + 1} = \frac{y^3}{y^2 + 1}$$

□

Solution— The proof that is presented shows that, for any given real number y , one can construct a x that satisfies the equation. But this is not what the statement of the theorem says. The theorem says that there is one particular x that works for all y .

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x, y)$ is **not** the same as $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, P(x, y)$.

Question 23— Consider the following incorrect theorem:

Theorem: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\cup \mathcal{F}$ and $\cup \mathcal{G}$ are disjoint, then so are $\mathcal{F}$ and $\mathcal{G}$.

What's wrong with the following proof of the theorem?

Proof: Suppose $\cup \mathcal{F}$ and $\cup \mathcal{G}$ are disjoint. Suppose $\mathcal{F}$ and $\mathcal{G}$ are not disjoint. Then we can choose some set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$.

Since $A \in \mathcal{F}$, by exercise 8, $A \subseteq \cup \mathcal{F}$, so every element of A is in $\cup \mathcal{F}$. Similarly, since $A \in \mathcal{G}$, every element of A is in $\cup \mathcal{G}$. But then every element of A is in both $\cup \mathcal{F}$ and $\cup \mathcal{G}$, and this is impossible since $\cup \mathcal{F}$ and $\cup \mathcal{G}$ are disjoint. Thus, we have reached a contradiction, so $\mathcal{F}$ and $\mathcal{G}$ must be disjoint.

□

Solution— This one is a tricky one, and a perfect occasion to remember that a statement of the form $\forall x \in A$ does not imply the existence of such an element.

- a) The proof looks right up until the sentence

But then every element of A is in both $\cup \mathcal{F}$ and $\cup \mathcal{G}$

It is very important to notice that $A \subseteq (\cup \mathcal{F}) \cap (\cup \mathcal{G})$ is equivalent to $\forall x \in A, x \in (\cup \mathcal{F}) \cap (\cup \mathcal{G})$.

This being true does not imply that such an element exists. In fact, it is possible that $A = \emptyset$, in which case the statement is vacuously true. Therefore, the proof is incorrect, as it is also assuming that A is nonempty.

- b) A counterexample can be produced by taking any disjoint and non-empty $\mathcal{F}$ and $\mathcal{G}$, and adding the empty set to both families. For example, let $\mathcal{F} = \{\emptyset, \{1\}\}$ and $\mathcal{G} = \{\emptyset, \{2\}\}$. Then $\cup \mathcal{F} = \{1\}$ and $\cup \mathcal{G} = \{2\}$ are disjoint, but $\mathcal{F}$ and $\mathcal{G}$ are not disjoint since they both contain the empty set.

3.4 Proofs Involving Conjunctions and Biconditionals

Question 2— Prove that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Solution— Suppose $A \subseteq B$ and $A \subseteq C$. We want to prove that $A \subseteq B \cap C$. To do this, we will assume that $x \in A$ and try to prove that $x \in B \cap C$. Since $A \subseteq B$, we can conclude that $x \in B$. Since $A \subseteq C$, we can conclude that $x \in C$. Therefore, since $x \in B$ and $x \in C$, we can conclude that $x \in B \cap C$. Thus, if $x \in A$ then $x \in B \cap C$, which proves that $A \subseteq B \cap C$.

Question 3— Suppose $A \subseteq B$. Prove that for every set C , $C \setminus B \subseteq C \setminus A$.

Solution— Suppose $A \subseteq B$. Let C be any set. We want to prove that $C \setminus B \subseteq C \setminus A$. To do this, we will assume that $x \in C \setminus B$ and try to prove that $x \in C \setminus A$. Since $x \in C \setminus B$, we know that $x \in C$ and $x \notin B$. Since $A \subseteq B$, we have that if $x \in A$, then $x \in B$. But we know that $x \notin B$, so it must be that $x \notin A$. Therefore, since $x \in C$ and $x \notin A$, we can conclude that $x \in C \setminus A$. Thus, if $x \in C \setminus B$ then $x \in C \setminus A$, which proves that $C \setminus B \subseteq C \setminus A$.

Question 4— Prove that if $A \subseteq B$ and $A \not\subseteq C$, then $B \not\subseteq C$.

Solution— Suppose $A \subseteq B$ and $A \not\subseteq C$. We want to prove that $B \not\subseteq C$. To do this, we will assume that $B \subseteq C$ and try to reach a contradiction. Since $A \subseteq B$ and $B \subseteq C$, we can conclude that $A \subseteq C$. But this contradicts our assumption that $A \not\subseteq C$. Therefore, our assumption that $B \subseteq C$ must be false, which proves that $B \not\subseteq C$.

Question 5— Prove that if $A \subseteq B \setminus C$ and $A \neq \emptyset$, then $B \not\subseteq C$.

Solution— Suppose $A \subseteq B \setminus C$ and $A \neq \emptyset$. We want to prove that $B \not\subseteq C$. To do this, we will assume that $B \subseteq C$ and try to reach a contradiction. Since $A \subseteq B$ and $B \subseteq C$, we can conclude that $A \subseteq C$. But this contradicts our assumption that $A \subseteq B \setminus C$, since if $A \subseteq B \setminus C$, then no element of A can be in C . Therefore, our assumption that $B \subseteq C$ must be false, which proves that $B \not\subseteq C$.

Question 9— Prove that if x and y are odd integers, then xy is odd.

Solution— Suppose x and y are odd integers. Then there exist integers k and l such that $x = 2k + 1$ and $y = 2l + 1$. We want to prove that xy is odd. Computing the product, we have:

$$xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

Since $2kl + k + l$ is an integer, we can conclude that xy is of the form $2m + 1$ for some integer m , which means xy is odd.

Question 10— Prove that for every integer n , n^3 is even iff n is even.

Solution— Suppose n is an integer. We want to prove that n^3 is even iff n is even.

First, suppose n is even. Then there exists an integer k such that $n = 2k$. Computing the cube, we have:

$$n^3 = (2k)^3 = 8k^3 = 2(4k^3).$$

Since $4k^3$ is an integer, we can conclude that n^3 is of the form $2m$ for some integer m , which means n^3 is even.

Now, suppose n^3 is even. Then there exists an integer m such that $n^3 = 2m$. We want to prove that n is even. We will prove this by contraposition. Suppose n is odd. Then there exists an integer l such that $n = 2l + 1$. Computing the cube, we have:

$$n^3 = (2l + 1)^3 = 8l^3 + 12l^2 + 6l + 1 = 2(4l^3 + 6l^2 + 3l) + 1.$$

Since $4l^3 + 6l^2 + 3l$ is an integer, we can conclude that n^3 is of the form $2p + 1$ for some integer p , which means n^3 is odd. This contradicts our assumption that n^3 is even. Therefore, if n^3 is even then n must be even.

Combining both directions, we have shown that n^3 is even iff n is even.

Question 11— Consider the following theorem:

Theorem: Suppose m is an even integer and n is an odd integer. Then $n^2 - m^2 = n + m$.

a) What's wrong with the following proof of the theorem?

Proof: Since m is even, we can choose some integer k such that $m = 2k$. Similarly, since n is odd we have $n = 2k + 1$. Therefore

$$n^2 - m^2 = (2k + 1)^2 - (2k)^2 = 4k^2 + 4k + 1 - 4k^2 = 4k + 1 = (2k + 1) + (2k) = n + m.$$

□

b) Is the theorem correct? If so, prove it. If not, find a counterexample.

Solution—

a) The proof is incorrect because it assumes that the same integer k can be used to express both m and n . In general, if m is even and n is odd, there is no reason to assume that they can be

expressed in terms of the same integer k . Therefore, the proof is incorrect.

- b) The theorem is not correct. A counterexample is $m = 2$ and $n = 5$. Then m is even and n is odd, but $n^2 - m^2 = 25 - 4 = 21$, while $n + m = 5 + 2 = 7$. Therefore, $n^2 - m^2 \neq n + m$, which contradicts the conclusion of the theorem. Thus, the theorem is incorrect.

Question 25— Prove that for all integers a and b , there is an integer c such that $a \mid c$ and $b \mid c$.

Solution— Let a and b be integers. We want to find an integer c such that $a \mid c$ and $b \mid c$. A suitable choice is $c = ab$. Indeed, $a \mid ab$ since $ab = a \cdot b$, and $b \mid ab$ since $ab = b \cdot a$.

Question 26—

- a) Prove that for every integer n , $15 \mid n$ iff $3 \mid n$ and $5 \mid n$.
b) Prove that it is not true that for every integer n , $60 \mid n$ iff $6 \mid n$ and $10 \mid n$.

Solution—

- a) Suppose n is an integer. We want to prove that $15 \mid n$ iff $3 \mid n$ and $5 \mid n$.

First, suppose $15 \mid n$. Then there exists an integer k such that $n = 15k = 3(5k)$. Since $5k$ is an integer, we can conclude that $3 \mid n$. Similarly, we can write $n = 15k = 5(3k)$. Since $3k$ is an integer, we can conclude that $5 \mid n$. Therefore, if $15 \mid n$ then $3 \mid n$ and $5 \mid n$.

Now suppose $3 \mid n$ and $5 \mid n$. Then there exist integers k and l such that $n = 3k$ and $n = 5l$. We mimic what is done in the book in Theorem 3.4.6: notice that $15k = 5(3k) = 5n$ and $15l = 3(5l) = 3n$. Therefore, we can write

$$15(k - l) = 15k - 15l = 5n - 3n = 2n$$

Let $j = (k - l)$. We have that $15j = 2n$. Now, we do a similar thing:

$$15(l - j) = 15l - 15j = 3n - 2n = n$$

Thus, we have found an integer $l - j$ such that $15(l - j) = n$, which means that $15 \mid n$. Therefore, if $3 \mid n$ and $5 \mid n$, then $15 \mid n$.

- b) This statement is false. A counterexample is $n = 30$. Then $6 \mid 30$ and $10 \mid 30$, but $60 \nmid 30$. Therefore, it is not true that for every integer n , $60 \mid n$ iff $6 \mid n$ and $10 \mid n$.

It has not been said explicitly, but one key difference between the two items is that 3 and 5 are relatively prime, while 6 and 10 are not. This is why the first statement is true and the second statement is false.

3.5 Proofs Involving Disjunctions

Question 1— Suppose A , B , and C are sets. Prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Solution— Suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we have that $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup C$. If $x \in C$, then $x \in (A \cap B) \cup C$. Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Question 2— Suppose A, B , and C are sets. Prove that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

Solution— Let $x \in (A \cup B) \setminus C$. Then $x \notin C$ and either $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \setminus C)$. If $x \in B$, then $x \in B \setminus C$, so again $x \in A \cup (B \setminus C)$.

Question 3— Suppose A and B are sets. Prove that $A \setminus (A \setminus B) = A \cap B$.

Solution— Suppose $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. Since $x \notin A \setminus B$, we have that $x \notin A$ or $x \in B$. But since $x \in A$, we must have that $x \in B$. Therefore, $x \in A$ and $x \in B$, which means that $x \in A \cap B$.

Now suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in B$, we have that $x \notin A \setminus B$. Therefore, $x \in A$ and $x \notin A \setminus B$, which means that $x \in A \setminus (A \setminus B)$.

Question 4— Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Prove that $A \subseteq B$.

Solution— We can prove this by contraposition. Suppose $A \not\subseteq B$. Then there exists an element $x \in A$ such that $x \notin B$. We want to show that either $A \cap C \not\subseteq B \cap C$ or $A \cup C \not\subseteq B \cup C$.

Since $x \in A$ and $x \notin B$, we have $x \in A \setminus B$. If $x \in C$, then $x \in A \cap C$ and $x \notin B \cap C$, which means that $A \cap C \not\subseteq B \cap C$. If $x \notin C$, then $x \in A$ and $x \notin B$, which means that $x \in A \cup C$ and $x \notin B \cup C$, which means that $A \cup C \not\subseteq B \cup C$. Therefore, if $A \not\subseteq B$, then either $A \cap C \not\subseteq B \cap C$ or $A \cup C \not\subseteq B \cup C$. This proves the contrapositive, and thus proves that if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

Question 6— Suppose A, B , and C are sets. Prove that $A \cup C \subseteq B \cup C$ if and only if $A \setminus C \subseteq B \setminus C$.

Solution— First suppose $A \cup C \subseteq B \cup C$. We want to prove that $A \setminus C \subseteq B \setminus C$. To do this, we will assume that $x \in A \setminus C$ and try to prove that $x \in B \setminus C$. Since $x \in A \setminus C$, we know that $x \in A$ and $x \notin C$. Since $x \in A$, we can conclude that $x \in A \cup C$. Since $A \cup C \subseteq B \cup C$, we can conclude that $x \in B \cup C$. Since $x \notin C$, we can conclude that $x \in B$. Therefore, since $x \in B$ and $x \notin C$, we can conclude that $x \in B \setminus C$. Thus, if $x \in A \setminus C$ then $x \in B \setminus C$, which proves that $A \setminus C \subseteq B \setminus C$. Now suppose $A \setminus C \subseteq B \setminus C$. We want to prove that $A \cup C \subseteq B \cup C$. To do this, we will assume that $x \in A \cup C$ and try to prove that $x \in B \cup C$. Since $x \in A \cup C$, we know that either $x \in A$ or $x \in C$. If $x \in C$, then we can conclude that $x \in B \cup C$. If $x \in A$, then we have two cases: either $x \in C$ or $x \notin C$. If $x \in C$, then we can conclude that $x \in B \cup C$. If $x \notin C$, then we can conclude that $x \in A \setminus C$. Since $A \setminus C \subseteq B \setminus C$, we can conclude that $x \in B \setminus C$. Therefore, since $x \in B$ and $x \notin C$, we can conclude that $x \in B$. Thus, if $x \in A$, then either $x \in B$ or $x \in C$, which means that if $x \in A$, then $x \in B \cup C$. Therefore, if $x \in A$ or $x \in C$, then $x \in B$ or $x \in C$, which means that if

$x \in A \cup C$, then $x \in B \cup C$. Thus, if $A \setminus C \subseteq B \setminus C$, then $A \cup C \subseteq B \cup C$.

Question 9— Suppose x and y are real numbers and $x \neq 0$. Prove that $y + \frac{1}{x} = 1 + \frac{y}{x}$ if and only if either $x = 1$ or $y = 1$.

Solution— Since $x \neq 0$, multiply the equation by x :

$$xy + 1 = x + y \iff xy - x - y + 1 = 0 \iff (x - 1)(y - 1) = 0.$$

By the zero-product property, this is equivalent to $x = 1$ or $y = 1$.

Question 12—

- a) Prove that for all real numbers a and b , $|a| \leq b$ if and only if $-b \leq a \leq b$.
- b) Prove that for any real number x , $-|x| \leq x \leq |x|$. (Hint: Use part (a).)
- c) Prove that for all real numbers x and y , $|x + y| \leq |x| + |y|$. (This is called the *triangle inequality*. One way to prove this is to combine parts (a) and (b), but you can also do it by considering a number of cases.)

Solution—

- a) For $b \geq 0$, $|a| \leq b$ means either $-b \leq a \leq b$. If $b < 0$, both statements are false. Thus $|a| \leq b \iff -b \leq a \leq b$.
- b) Apply part (a) with $a = x$ and $b = |x|$. Since $|x| \leq |x|$, we obtain $-|x| \leq x \leq |x|$.
- c) By part (b), $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying part (a) with $a = x + y$ and $b = |x| + |y|$ yields $|x + y| \leq |x| + |y|$.

Question 13— Prove that for every integer x , $x^2 + x$ is even.

Solution— Suppose x is an integer. We want to prove that $x^2 + x$ is even. We will prove this by considering two cases: when x is even and when x is odd.

Case 1: Suppose x is even. Then there exists an integer k such that $x = 2k$. Computing $x^2 + x$, we have:

$$x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k).$$

Since $2k^2 + k$ is an integer, we can conclude that $x^2 + x$ is of the form $2m$ for some integer m , which means $x^2 + x$ is even.

Case 2: Suppose x is odd. Then there exists an integer l such that $x = 2l + 1$. Computing $x^2 + x$, we have:

$$x^2 + x = (2l + 1)^2 + (2l + 1) = (4l^2 + 4l + 1) + (2l + 1) = 4l^2 + 6l + 2 = 2(2l^2 + 3l + 1).$$

Since $2l^2 + 3l + 1$ is an integer, we can conclude that $x^2 + x$ is of the form $2m$ for some integer m , which means $x^2 + x$ is even.

Since in both cases we have shown that $x^2 + x$ is even, we can conclude that for every integer x , $x^2 + x$ is even.

Question 14— Prove that for every integer x , the remainder when x^4 is divided by 8 is either 0 or 1.

Solution— Suppose x is an integer. We want to prove that the remainder when x^4 is divided by 8 is either 0 or 1. We will prove this by considering two cases: when x is even and when x is odd.

Case 1: Suppose x is even. Then there exists an integer k such that $x = 2k$. Computing x^4 , we have:

$$x^4 = (2k)^4 = 16k^4 = 8(2k^4).$$

Since $2k^4$ is an integer, we can conclude that x^4 is divisible by 8, which means the remainder is 0.

Case 2: Suppose x is odd. Then there exists an integer l such that $x = 2l + 1$. Computing x^4 , we have:

$$x^4 = (2l + 1)^4.$$

Expanding this expression, we get:

$$x^4 = 16l^4 + 32l^3 + 24l^2 + 8l + 1.$$

Factoring out 8, we have:

$$x^4 = 8(2l^4 + 4l^3 + 3l^2 + l) + 1.$$

Since $2l^4 + 4l^3 + 3l^2 + l$ is an integer, we can conclude that x^4 is of the form $8m + 1$ for some integer m , which means the remainder is 1.

Since in both cases we have shown that the remainder when x^4 is divided by 8 is either 0 or 1, we can conclude that for every integer x , the remainder when x^4 is divided by 8 is either 0 or 1.

Other recommended exercises: 26, 27, 28, 29, 30, 31

4 Relations

4.1 Ordered Pairs and Cartesian Products

Question 7— If A has m elements and B has n elements, how many elements does $A \times B$ have?

Solution— If A has m elements and B has n elements, then the Cartesian product $A \times B$ consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. For each of the m elements in A , there are n choices for the element in B . Therefore, the total number of ordered pairs is given by multiplying the number of choices for each element:

$$|A \times B| = m \cdot n.$$

Question 10— Prove that for any sets A , B , C , and D , if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint, or B and D are disjoint.

Solution— We will prove this by contradiction. Suppose $A \times B$ and $C \times D$ are disjoint, but neither A and C are disjoint nor B and D are disjoint. Then there exists an element $a \in A \cap C$ and an element $b \in B \cap D$. Therefore, the ordered pair (a, b) is in both $A \times B$ and $C \times D$, which contradicts the assumption that $A \times B$ and $C \times D$ are disjoint. Therefore, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Question 12— Consider the following theorem.

Theorem: For any sets A , B , C , and D , if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof: Suppose $A \times B \subseteq C \times D$. Let a be an arbitrary element of A and let b be an arbitrary element of B . Then $(a, b) \in A \times B$, so since $A \times B \subseteq C \times D$, we have $(a, b) \in C \times D$. Therefore, $a \in C$ and $b \in D$. Since a and b were arbitrary elements of A and B , respectively, this shows that $A \subseteq C$ and $B \subseteq D$.

□

Solution— There is one subtle extra assumption that the proof is making: the existence of a pair $(a, b) \in A \times B$. The proof starts correctly by taking arbitrary elements of A and B separately. This is the traditional way to go, as “an arbitrary” is not assuming such an element exists.

However, when putting them together into a pair (a, b) , the proof is implicitly assuming that A and B are non-empty. Notice the subtle difference with “taking an arbitrary element of $A \times B$ ”, which would be a valid approach.

To show that the proof is not correct, and that actually the result is false, we can create a somewhat convoluted counterexample. The trick is to play with a very particular case: when sets are empty.

Take $A = \emptyset$, $B = \{1\}$, $C = \mathbb{N}$, $D = \emptyset$.

Notice that $A \times B = C \times D = \emptyset$. Thus, trivially, $A \times B \subseteq C \times D$. However, $A \subseteq C$ is true, but $B \subseteq D$ is false. Therefore, the theorem is false.

4.2 Relations

Recall that a relation R from a set A to a set B is a subset of the Cartesian product $A \times B$. The domain of R , denoted $Dom(R)$, is the set of all elements $a \in A$ such that there exists an element $b \in B$ with $(a, b) \in R$. The range of R , denoted $Ran(R)$, is the set of all elements $b \in B$ such that there exists an element $a \in A$ with $(a, b) \in R$.

Given a relation R from a set A to a set B and a relation S from a set B to a set C , the composition of R and S , denoted $S \circ R$, is the relation from A to C defined by $(a, c) \in S \circ R$ if and only if there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Given a relation R from a set A to a set B , the inverse of R , denoted R^{-1} , is the relation from B to A defined by $(b, a) \in R^{-1}$ if and only if $(a, b) \in R$.

A relation R from a set A to itself is called a relation on A . A relation R on a set A is called reflexive if $(a, a) \in R$ for all $a \in A$, symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$, and transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Now you are more than ready to tackle the following exercises. Good luck.

Question 1— Find the domains and ranges of the following relations.

- a) $\{(p, q) \in P \times P \mid \text{the person } p \text{ is a parent of the person } q\}$, where P is the set of all living people.
- b) $\{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$.

Solution—

- a) For a pair (p, q) to be in the relation, p must be a parent of q . Therefore, the domain is the set of all living people who are parents (and whose children are also living), and the range is the set of all living people who have living parents.
- b) For a pair (x, y) to be in the relation, y must be greater than x^2 . Therefore, the domain is $\mathbb{R}$, as there is no particular constraint on x , and the range is $(0, \infty)$, because if we take $y \in (-\infty, 0]$, we will not be able to find any x such that $y > x^2$.

Question 2— Find the domains and ranges of the following relations.

- a) $\{(p, q) \in P \times P \mid \text{the person } p \text{ is a brother of the person } q\}$, where P is the set of all living people.
- b) $\{(x, y) \in \mathbb{R}^2 \mid y^2 = 1 - \frac{2}{x^2+1}\}$.

Solution—

- a) For a pair (p, q) to be in the relation, p must be a brother of q . Therefore, the domain is the set of all living men who are brothers (and whose siblings are also living), and the range is the set of all living people who have living brothers.
- b) We can rewrite the equation as $y^2 = \frac{x^2-1}{x^2+1}$. Notice that there will exist values of y satisfying this only if $x^2 - 1 \geq 0$, which means $x^2 \geq 1$. Therefore, the domain is $(-\infty, -1] \cup [1, \infty)$. To get the range, write $y^2 = (x^2 - 1)/(x^2 + 1)$. The right-hand side takes every value in $[0, 1)$, so y takes every value in $(-1, 1)$. Therefore, the range of the relation is $(-1, 1)$.

Question 8— Suppose R is a relation from A to B and S is a relation from B to C

- a) Prove that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$.
- b) Prove that if $\text{Ran}(R) \subseteq \text{Dom}(S)$, then $\text{Dom}(S \circ R) = \text{Dom}(R)$.
- c) Formulate and prove similar theorems for the range of $S \circ R$.

Solution—

- a) Recall that $S \circ R$ is a relation from A to C defined by $(a, c) \in S \circ R$ if and only if there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Suppose $a \in \text{Dom}(S \circ R)$. Then there exists $c \in C$ such that $(a, c) \in S \circ R$. By the definition of composition, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $(a, b) \in R$, we

have that $a \in \text{Dom}(R)$. Therefore, if $a \in \text{Dom}(S \circ R)$, then $a \in \text{Dom}(R)$, which proves that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$.

- b) Suppose $\text{Ran}(R) \subseteq \text{Dom}(S)$. We want to prove that $\text{Dom}(S \circ R) = \text{Dom}(R)$. We already know from part (a) that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$, so we only need to prove that $\text{Dom}(R) \subseteq \text{Dom}(S \circ R)$.

Suppose $a \in \text{Dom}(R)$. Then there exists $b \in B$ such that $(a, b) \in R$. Notice that $b \in \text{Ran}(R)$, and since $\text{Ran}(R) \subseteq \text{Dom}(S)$, we have that $b \in \text{Dom}(S)$. Therefore, there exists $c \in C$ such that $(b, c) \in S$. By the definition of composition, we have that $(a, c) \in S \circ R$. Therefore, $a \in \text{Dom}(S \circ R)$. Thus, if $a \in \text{Dom}(R)$, then $a \in \text{Dom}(S \circ R)$, which proves that $\text{Dom}(R) \subseteq \text{Dom}(S \circ R)$.

- c) First, $\text{Ran}(S \circ R) \subseteq \text{Ran}(S)$. If $c \in \text{Ran}(S \circ R)$, then there is some $a \in A$ such that $(a, c) \in S \circ R$. Hence, for some $b \in B$, we have $(a, b) \in R$ and $(b, c) \in S$, so $c \in \text{Ran}(S)$.

If $\text{Dom}(S) \subseteq \text{Ran}(R)$, then $\text{Ran}(S \circ R) = \text{Ran}(S)$. The inclusion $\text{Ran}(S \circ R) \subseteq \text{Ran}(S)$ was just proved. Conversely, let $c \in \text{Ran}(S)$. Then there is some $b \in B$ such that $(b, c) \in S$, so $b \in \text{Dom}(S)$. By the hypothesis, $b \in \text{Ran}(R)$, and hence there is some $a \in A$ such that $(a, b) \in R$. Therefore $(a, c) \in S \circ R$, so $c \in \text{Ran}(S \circ R)$.

Question 9— Suppose R and S are relations from A to B . Must the following statements be true? Justify your answers with proofs or counterexamples.

a) $R \subseteq \text{Dom}(R) \times \text{Ran}(R)$

b) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Solution—

- a) True by definition of domain and range. However, the converse is not true. That is, $\text{Dom}(R) \times \text{Ran}(R) \subseteq R$ is not true in general.
- b) True. Suppose $R \subseteq S$. Then for every $(a, b) \in R$, we have $(a, b) \in S$. By the definition of inverse relations, this means that $(b, a) \in R^{-1}$ and $(b, a) \in S^{-1}$. Therefore, $R^{-1} \subseteq S^{-1}$.
- c) True. If $(b, a) \in (R \cup S)^{-1}$, then $(a, b) \in R \cup S$, so $(b, a) \in R^{-1} \cup S^{-1}$. Conversely, if $(b, a) \in R^{-1} \cup S^{-1}$, then $(a, b) \in R \cup S$, so $(b, a) \in (R \cup S)^{-1}$. Thus the two relations are equal.

Question 10— Suppose R is a relation from A to B and S is a relation from B to C . Prove that $S \circ R = \emptyset$ if and only if $\text{Ran}(R)$ and $\text{Dom}(S)$ are disjoint.

Solution— Both directions will be proved by contraposition.

($\leftarrow$) Suppose $\text{Ran}(R)$ and $\text{Dom}(S)$ are not disjoint. Then there exists an element $b \in \text{Ran}(R) \cap \text{Dom}(S)$. By the definition of range, there exists $a \in A$ such that $(a, b) \in R$. By the definition of domain, there exists $c \in C$ such that $(b, c) \in S$. Therefore, by the definition of composition, we have that $(a, c) \in S \circ R$, which means that $S \circ R \neq \emptyset$.

($\rightarrow$) Suppose $S \circ R \neq \emptyset$. Then there exists an element $(a, c) \in S \circ R$. By the definition of composition, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Therefore, $b \in \text{Ran}(R)$ and $b \in \text{Dom}(S)$,

which means that $\text{Ran}(R)$ and $\text{Dom}(S)$ are not disjoint.

4.3 More About Relations

Question 11— Suppose R is a relation on A . Prove that if R is reflexive then $R \subseteq R \circ R$.

Solution— Suppose R is reflexive. We want to prove that $R \subseteq R \circ R$. To do this, we will assume that $(a, b) \in R$ and try to prove that $(a, b) \in R \circ R$. Since R is reflexive, we have that $(b, b) \in R$. Therefore, by the definition of composition, we have that $(a, b) \in R \circ R$. Thus, if $(a, b) \in R$, then $(a, b) \in R \circ R$, which proves that $R \subseteq R \circ R$.

Question 12— Suppose R is a relation on A . Prove that:

- a) If R is reflexive, so is R^{-1} .
- b) If R is symmetric, so is R^{-1} .
- c) If R is transitive, so is R^{-1} .

Solution—

- a) Suppose R is reflexive. Then for every $a \in A$, we have $(a, a) \in R$. By the definition of inverse relations, we have that $(a, a) \in R^{-1}$. Therefore, R^{-1} is reflexive.
- b) Suppose R is symmetric and consider any $(b, a) \in R^{-1}$. By definition of inverse relations, we have that $(a, b) \in R$. Since R is symmetric, we have that $(b, a) \in R$. Again, by definition of inverse relations, we have that $(a, b) \in R^{-1}$. Therefore, R^{-1} is symmetric.
- c) Suppose R is transitive. Suppose $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. By definition of inverse relations, we have that $(b, a) \in R$ and $(c, b) \in R$. Since R is transitive, we have that $(c, a) \in R$. Again, by definition of inverse relations, we have that $(a, c) \in R^{-1}$. Therefore, R^{-1} is transitive.

Question 13— Suppose R_1 and R_2 are relations on A . For each part, give either a proof or a counterexample to justify your answer.

- a) If R_1 and R_2 are reflexive, then $R_1 \cup R_2$ is reflexive.
- b) If R_1 and R_2 are symmetric, then $R_1 \cup R_2$ is symmetric.
- c) If R_1 and R_2 are transitive, then $R_1 \cup R_2$ is transitive.

Solution—

- a) True. Suppose R_1 and R_2 are reflexive. Then for every $a \in A$, we have $(a, a) \in R_1$ and $(a, a) \in R_2$. Therefore, $(a, a) \in R_1 \cup R_2$. Since this is true for every $a \in A$, we can conclude that $R_1 \cup R_2$ is reflexive.
- b) True. Suppose R_1 and R_2 are symmetric. Consider any $(a, b) \in R_1 \cup R_2$. Then either $(a, b) \in R_1$ or $(a, b) \in R_2$. If $(a, b) \in R_1$, then since R_1 is symmetric, we have that $(b, a) \in R_1$, which means that $(b, a) \in R_1 \cup R_2$. If $(a, b) \in R_2$, then since R_2 is symmetric, we have that $(b, a) \in R_2$, which means that $(b, a) \in R_1 \cup R_2$. Therefore, in either case, we have that

$(b, a) \in R_1 \cup R_2$. Since this is true for any $(a, b) \in R_1 \cup R_2$, we can conclude that $R_1 \cup R_2$ is symmetric.

- c) False. Let $A = \{1, 2, 3\}$, $R_1 = \{(1, 2)\}$, and $R_2 = \{(2, 3)\}$. Each relation is transitive, but $R_1 \cup R_2$ contains $(1, 2)$ and $(2, 3)$ and does not contain $(1, 3)$, so it is not transitive.

Question 14— Suppose R_1 and R_2 are relations on A . For each part, give either a proof or a counterexample to justify your answer.

- a) If R_1 and R_2 are reflexive, then $R_1 \cap R_2$ is reflexive.
- b) If R_1 and R_2 are symmetric, then $R_1 \cap R_2$ is symmetric.
- c) If R_1 and R_2 are transitive, then $R_1 \cap R_2$ is transitive.

Solution—

- a) True. Suppose R_1 and R_2 are reflexive. Then for every $a \in A$, we have $(a, a) \in R_1$ and $(a, a) \in R_2$. Therefore, $(a, a) \in R_1 \cap R_2$. Since this is true for every $a \in A$, we can conclude that $R_1 \cap R_2$ is reflexive.
- b) True. Suppose R_1 and R_2 are symmetric. Consider any $(a, b) \in R_1 \cap R_2$. Then $(a, b) \in R_1$ and $(a, b) \in R_2$. Since both relations are symmetric, we have that $(b, a) \in R_1$ and $(b, a) \in R_2$. Therefore, $(b, a) \in R_1 \cap R_2$. Since this is true for any $(a, b) \in R_1 \cap R_2$, we can conclude that $R_1 \cap R_2$ is symmetric.
- c) True. Suppose R_1 and R_2 are transitive. Consider any $(a, b), (b, c) \in R_1 \cap R_2$. Then $(a, b) \in R_1$ and $(b, c) \in R_1$, and also $(a, b) \in R_2$ and $(b, c) \in R_2$. Since both relations are transitive, we have that $(a, c) \in R_1$ and $(a, c) \in R_2$. Therefore, $(a, c) \in R_1 \cap R_2$. Since this is true for any $(a, b), (b, c) \in R_1 \cap R_2$, we can conclude that $R_1 \cap R_2$ is transitive.

Question 15— Suppose R_1 and R_2 are relations on A . For each part, give either a proof or a counterexample to justify your answer.

- a) If R_1 and R_2 are reflexive, then $R_1 \setminus R_2$ is reflexive.
- b) If R_1 and R_2 are symmetric, then $R_1 \setminus R_2$ is symmetric.
- c) If R_1 and R_2 are transitive, then $R_1 \setminus R_2$ is transitive.

Solution—

- a) False. Take $A = \{1, 2\}$, $R_1 = A \times A$ and $R_2 = \{(1, 1), (2, 2)\}$. Then $R_1 \setminus R_2 = \{(1, 2), (2, 1)\}$, which is not reflexive.
- b) True. If $(a, b) \in R_1 \setminus R_2$, then $(a, b) \in R_1$ and $(a, b) \notin R_2$. Since R_1 is symmetric, we have that $(b, a) \in R_1$. Since R_2 is symmetric, we have that $(b, a) \notin R_2$, otherwise $(a, b) \in R_2$. Therefore, $(b, a) \in R_1 \setminus R_2$. Since this is true for any $(a, b) \in R_1 \setminus R_2$, we can conclude that $R_1 \setminus R_2$ is symmetric.
- c) False. Take $A = \{1, 2, 3\}$, $R_1 = \{(1, 2), (2, 3), (1, 3)\}$ and $R_2 = \{(1, 3)\}$. Then $R_1 \setminus R_2 = \{(1, 2), (2, 3)\}$, which is not transitive.

Question 16— Suppose R and S are transitive relations on A . Prove that if $S \circ R \subseteq R \circ S$, then $R \circ S$ is transitive.

Solution— Suppose $(a, b), (b, c) \in R \circ S$. Choose $u, v \in A$ such that $(a, u) \in S$, $(u, b) \in R$, $(b, v) \in S$, and $(v, c) \in R$. By transitivity of R , $(u, c) \in R$. Hence $(b, c) \in S \circ R$. The hypothesis gives $(b, c) \in R \circ S$, but we need the middle link in the other order. Instead, from $(u, b) \in R$ and $(b, v) \in S$, we have $(u, v) \in S \circ R \subseteq R \circ S$. Choose w with $(u, w) \in S$ and $(w, v) \in R$. Since S is transitive, $(a, w) \in S$, and since R is transitive, $(w, c) \in R$. Therefore $(a, c) \in R \circ S$, proving transitivity.

4.4 Ordering Relations

A relation R on a set A is called a partial order if it is reflexive, antisymmetric, and transitive. A partial order R on a set A is called a total order if for every pair of elements $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$. In other words, a total order is a partial order in which every pair of elements is comparable.

Question 1— In each case, say whether or not R is a partial order on A . If so, is it a total order?

- a) $A = \{a, b, c\}$, $R = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.
- b) $A = \mathbb{R}$, $R = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq |y|\}$.
- c) $A = \mathbb{R}$, $R = \{(x, y) \in \mathbb{R}^2 \mid |x| < |y| \text{ or } x = y\}$.

Solution—

- a) R is a partial order on A . It is reflexive, antisymmetric, and transitive. However, it is not a total order because not all pairs of elements in A are comparable. For example, $(a, c) \notin R$ and $(c, a) \notin R$.
- b) R is not a partial order on A . It is not antisymmetric, as for example $(1, -1) \in R$ and $(-1, 1) \in R$, but $1 \neq -1$.
- c) R is a partial order. One can prove by cases the antisymmetry and transitivity. It is not a total order, as for example $(1, -1) \notin R$ and $(-1, 1) \notin R$.

Question 2— In each case, say whether or not R is a partial order on A . If so, is it a total order?

- a) Let A be the set of all words of English, and $R \subset A \times A$ be all the pairs (x, y) such that the word y occurs at least as late in alphabetical order as the word x .
- b) Let A be the set of all words of English, $R \subset A \times A$ be all the pairs (x, y) such that the first letter of the word y occurs at least as late in the alphabet as the first letter of the word x .
- c) Let A be the set of all countries in the world, and $R \subset A \times A$ be all the pairs (x, y) such that the population of the country y is at least as large as the population of the country x .

Solution—

- a) R is a partial order on A . It is also a total order, as any two words can be compared in alphabetical order.

- b) R is not an order, as it is not antisymmetric. For example, the words “cat” and “caterpillar” both start with the letter “c”, so $(\text{cat}, \text{caterpillar}) \in R$ and $(\text{caterpillar}, \text{cat}) \in R$, but $\text{cat} \neq \text{caterpillar}$.
- c) R is not an order on A since, even if highly unlikely, there can be countries with the same population. If country x and country y have the same population, then $(x, y) \in R$ and $(y, x) \in R$, but $x \neq y$. If we add the (very realistic) assumption that no two countries have the same population, then the set of countries can be identified with a finite subset of integers with the usual order, and thus R is a total order.

Question 5— Suppose R is a partial order on A and $B \subseteq A$. Prove that $R \cap (B \times B)$ is a partial order on B .

Solution— Consider any $b \in B$. Since $B \subseteq A$, we have that $b \in A$. Since R is a partial order on A , we have that $(b, b) \in R$. Therefore, $(b, b) \in R \cap (B \times B)$, which proves that $R \cap (B \times B)$ is reflexive. Now suppose $(b_1, b_2) \in R \cap (B \times B)$ and $(b_2, b_1) \in R \cap (B \times B)$. Then $(b_1, b_2) \in R$ and $(b_2, b_1) \in R$. Since R is a partial order on A , we have that $b_1 = b_2$. Therefore, $R \cap (B \times B)$ is antisymmetric. Finally, suppose $(b_1, b_2) \in R \cap (B \times B)$ and $(b_2, b_3) \in R \cap (B \times B)$. Then $(b_1, b_2) \in R$ and $(b_2, b_3) \in R$. Since R is a partial order on A , we have that $(b_1, b_3) \in R$. Therefore, $(b_1, b_3) \in R \cap (B \times B)$, which proves that $R \cap (B \times B)$ is transitive.

Question 6— Suppose R_1 and R_2 are partial orders on A . For each part, give either a proof or a counterexample to justify your answer.

- a) $R_1 \cap R_2$ is a partial order on A .
- b) $R_1 \cup R_2$ is a partial order on A .

Solution—

$R_1 \cap R_2$ is a partial order on A . The proof is similar to the proof of Question 5.

$R_1 \cup R_2$ is not necessarily a partial order. Let $A = \{1, 2, 3\}$, $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$, and $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$. Both are partial orders, but their union contains $(1, 2)$ and $(2, 3)$ without containing $(1, 3)$, so it is not transitive.

Question 10— Suppose R is a partial order on A . For each $x \in A$, let $P_x = \{a \in A \mid aRx\}$. Prove that $\forall x \in A, \forall y \in A, (xRy \leftrightarrow P_x \subseteq P_y)$.

Solution— Consider arbitrary $x, y \in A$. We want to prove that $xRy \leftrightarrow P_x \subseteq P_y$.

($\rightarrow$) Suppose xRy . Consider any $a \in P_x$. Then aRx . Since R is transitive, we have that aRy . Therefore, $a \in P_y$. Since this is true for any $a \in P_x$, we can conclude that $P_x \subseteq P_y$.

($\leftarrow$) Suppose $P_x \subseteq P_y$. Since R is reflexive, we have that xRx , which means that $x \in P_x$. Since $P_x \subseteq P_y$, we have that $x \in P_y$, which means that xRy .

Question 11— Let D be the divisibility relation on $\mathbb{Z}$, i.e., xDy if and only if x divides y . Let $B = \{x \in$

$\mathbb{Z} \mid x > 1\}$. Does B have any minimal elements? If so, what are they? Does B have a smallest element? If so, what is it?

Solution— It is worth recalling the definitions of minimal and smallest elements. An element $m \in B$ is a minimal element if there is no $b \in B$ such that bDm and $b \neq m$. An element $s \in B$ is a smallest element if for all $b \in B$, we have sDb .

In this case, we are looking for minimal elements, that is, elements $m \in B$ such that there is no $b \in B$ with bDm and $b \neq m$. The only elements in B that are not divisible by any other element in B are the prime numbers. Therefore, the minimal elements of B are the prime numbers.

Notice that there is no smallest element in B . A smallest element would be a divisor of all other positive integers in B . The only candidate we have is 1, but $1 \notin B$. Therefore, B does not have a smallest element.

4.5 Equivalence Relations

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. If R is an equivalence relation on A , then for each $a \in A$, the equivalence class of a is defined as $[a] = \{b \in A \mid aRb\}$. The set of all equivalence classes of R is denoted by $A/R = \{[a] \mid a \in A\}$. This family of sets is a partition of A , meaning that the equivalence classes are disjoint and their union is A .

Question 3— Let W be the set of all words in the English language. Which of the following relations on W are equivalence relations? For those that are equivalence relations, what are the equivalence classes?

- a) xRy if and only if x and y start with the same letter.
- b) xRy if and only if x and y have at least one letter in common.
- c) xRy if and only if x and y have the same number of letters.

Solution—

- a) This relation is an equivalence relation. It is reflexive, symmetric, and transitive. The equivalence classes are the sets of words that start with the same letter. In other words, the equivalence classes can be matched to the letters of the alphabet.
- b) This relation is not an equivalence relation. It is reflexive and symmetric, but it is not transitive. For example, the words $x = \text{"cat"}$ and $z = \text{"red"}$ have no letters in common, but both have letters in common with the word $y = \text{"car"}$. Therefore, xRy and yRz do not imply xRz .
- c) This relation is an equivalence relation. It is reflexive, symmetric, and transitive. The equivalence classes are the sets of words that have the same number of letters. In other words, the equivalence classes can be matched to the positive integers.

Question 4— Which of the following relations on $\mathbb{R}$ are equivalence relations? For those that are equivalence relations, what are the equivalence classes?

- a) xRy if and only if $x - y \in \mathbb{N}$.
- b) xSy if and only if $x - y \in \mathbb{Q}$.
- c) xTy if and only if $\exists n \in \mathbb{Z} \mid y = x \cdot 10^n$.

Solution—

- a) This relation is not an equivalence relation because it is not symmetric. For example, $2R1$ because $2 - 1 = 1 \in \mathbb{N}$, but $1R2$ is not true because $1 - 2 = -1 \notin \mathbb{N}$.
- b) This relation is an equivalence relation. It is reflexive, symmetric, and transitive because the addition of two rational numbers is a rational number. To better understand the equivalence classes, let us think of the equivalence class of a rational number q . As $q - 0 \in \mathbb{Q}$, we have that $q \in [0]$. Therefore, $[0] = \mathbb{Q}$. Now consider the equivalence class of an irrational number r . As $r - 0 \notin \mathbb{Q}$, we have that $r \notin [0]$. In this case, we can describe the equivalence class as $[r] = r + \mathbb{Q}$.

Not all irrational numbers have their own equivalence class. For example, consider the irrational numbers $r_1 = \sqrt{2}$ and $r_2 = 1 + \sqrt{2}$.

- c) This relation is an equivalence relation, and that is easy to prove using the definition. To understand the equivalence classes, we can take a rational number like 1.234, and think of all the numbers that can be obtained by multiplying it by 10^n for some integer n . This gives us the equivalence class $[1.234] = \{\dots, 0.01234, 0.1234, 1.234, 12.34, 123.4, 1234, \dots\}$. More generally, for $x \neq 0$, $[x] = \{x 10^n : n \in \mathbb{Z}\}$, while $[0] = \{0\}$. Reflexivity, symmetry, and transitivity follow respectively from exponents 0, $-n$, and $n + m$.

Question 10— Let m be a positive integer. The relation C_m is defined on $\mathbb{Z}$ by aC_mb if and only if m divides $a - b$. In other words, aC_mb if and only if there exists an integer k such that $a - b = km$. Another way to say this is that aC_mb if and only if a and b have the same remainder when divided by m .

- a) Show that C_m is an equivalence relation on $\mathbb{Z}$ by proving that it is reflexive, symmetric, and transitive.
- b) Find all the equivalence classes for C_2 and C_3 . How many equivalence classes are there in each case? In general, how many equivalence classes do you think there are for C_m ?

Solution—

- a) (i) (*reflexive:*) For any integer a , we have that $a - a = 0$, and m divides 0. Therefore, aC_ma for all integers a , which proves that C_m is reflexive.
- (ii) (*symmetric:*) Suppose aC_mb . Then m divides $a - b$, so there exists an integer k such that $a - b = km$. This means $b - a = -km$, and since m divides km , we have that m divides $b - a$. Therefore, bC_ma .
- (iii) (*transitive:*) Suppose aC_mb and bC_mc . Then m divides $a - b$ and m divides $b - c$. So there exist integers k_1 and k_2 such that $a - b = k_1m$ and $b - c = k_2m$. Adding these equations, we get $(a - b) + (b - c) = k_1m + k_2m$, which simplifies to $a - c = (k_1 + k_2)m$. Since the sum of integers is an integer, we have that m divides $a - c$, which means that aC_mc .
- b) For C_2 , the equivalence classes are:
- $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$ (even integers)
 - $[1] = \{\dots, -3, -1, 1, 3, 5, \dots\}$ (odd integers)

There are 2 equivalence classes for C_2 .

For C_3 , the equivalence classes are:

- $[0] = \{\dots, -6, -3, 0, 3, 6, \dots\}$
- $[1] = \{\dots, -5, -2, 1, 4, 7, \dots\}$
- $[2] = \{\dots, -4, -1, 2, 5, 8, \dots\}$

There are 3 equivalence classes for C_3 .

In general, for C_m , there are m equivalence classes corresponding to the remainders when dividing by m : $[0], [1], [2], \dots, [m-1]$.

Question 11— Prove that for every integer n , either $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$.

Solution— We split the proof into cases based on the parity of n .

Case 1: n is even. Then $n = 2k$ for some integer k . Therefore, $n^2 = (2k)^2 = 4k^2$, which means that $n^2 \equiv 0 \pmod{4}$.

Case 2: n is odd. Then $n = 2k + 1$ for some integer k . Therefore, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, which means that $n^2 \equiv 1 \pmod{4}$.

In both cases, we have shown that n^2 is congruent to either 0 or 1 modulo 4.

Question 12— Let m be a positive integer. Prove that for all integers a, b, c , and d , if $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $a + b \equiv c + d \pmod{m}$ and $ab \equiv cd \pmod{m}$.

Solution— Let $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. Suppose $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$. Then there exist integers k_1 and k_2 such that $a - c = k_1m$ and $b - d = k_2m$.

For the sum, we have:

$$(a + b) - (c + d) = (a - c) + (b - d) = k_1m + k_2m = (k_1 + k_2)m$$

This shows that $a + b \equiv c + d \pmod{m}$.

For the product, we have:

$$ab - cd = a(b - d) + d(a - c)$$

Since $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$, we have $a - c = k_1m$ and $b - d = k_2m$. Therefore:

$$ab - cd = a(k_2m) + d(k_1m) = (ak_2 + dk_1)m$$

This shows that $ab \equiv cd \pmod{m}$.

Question 13— Recall that given a set A and an equivalence relation R on A , the equivalence class of an element $x \in A$ is defined as $[x]_R = \{y \in A \mid xRy\}$.

Suppose R is an equivalence relation on A and $B \subseteq A$. Let $S = R \cap (B \times B)$.

- Prove that S is an equivalence relation on B .
- Prove the following relationship between equivalence classes: for all $x \in B$, $[x]_S = [x]_R \cap B$.

Solution—

- a) The proof that S is an equivalence relation on B is similar to the proof of Question 5. We need to show that S is reflexive, symmetric, and transitive. Since R is an equivalence relation on A , it is reflexive, symmetric, and transitive. Therefore, for any $b \in B$, we have $(b, b) \in R$, which means $(b, b) \in S$. This proves that S is reflexive. For symmetry, if $(b_1, b_2) \in S$, then $(b_1, b_2) \in R$, which means $(b_2, b_1) \in R$, and since $b_1, b_2 \in B$, we have $(b_2, b_1) \in S$. For transitivity, if $(b_1, b_2) \in S$ and $(b_2, b_3) \in S$, then $(b_1, b_2) \in R$ and $(b_2, b_3) \in R$, which means $(b_1, b_3) \in R$, and since $b_1, b_3 \in B$, we have $(b_1, b_3) \in S$. Therefore, S is an equivalence relation on B .
- b) Consider any $x \in B$. We want to prove that $[x]_S = [x]_R \cap B$.
- $(\subseteq)$ Suppose $y \in [x]_S$. Then $(x, y) \in S$, which means $(x, y) \in R$ and $y \in B$. Therefore, $y \in [x]_R$ and $y \in B$, which means $y \in [x]_R \cap B$.
- $(\supseteq)$ Suppose $y \in [x]_R \cap B$. Then $(x, y) \in R$ and $y \in B$. Therefore, $(x, y) \in S$, which means $y \in [x]_S$.
- Since both inclusions hold, we conclude that $[x]_S = [x]_R \cap B$.

5 Functions

A function from a set A to a set B is a relation $f \subseteq A \times B$ such that for each $a \in A$, there exists exactly one $b \in B$ such that $(a, b) \in f$. We write $f : A \rightarrow B$ to denote that f is a function from A to B . If $(a, b) \in f$, we write $f(a) = b$. The set A is called the domain of the function, and the set B is called the codomain. The range of the function is the set $\{b \in B \mid \exists a \in A, f(a) = b\}$.

5.1 Functions

Question 3—

- a) Let $A = \{a, b, c\}$, $B = \{a, b\}$, and $f = \{(a, b), (b, b), (c, a)\}$. Then $f : A \rightarrow B$. What are $f(a)$, $f(b)$, and $f(c)$?
- b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by the formula $f(x) = x^2 - 2x$. What is $f(2)$?
- c) Let $f = \{(x, n) \in \mathbb{R} \times \mathbb{Z} \mid n \leq x < n + 1\}$. Then $f : \mathbb{R} \rightarrow \mathbb{Z}$. What is $f(\pi)$? What is $f(-\pi)$?

Solution—

- a) We have $f(a) = b$, $f(b) = b$, and $f(c) = a$.
- b) We have $f(2) = 2^2 - 2 \cdot 2 = 4 - 4 = 0$.
- c) We have $f(\pi) = 3$ and $f(-\pi) = -4$. This function is the floor function, which returns the greatest integer less than or equal to x .

Question 7— Suppose $f : A \rightarrow B$ and $C \subseteq A$. The set $f \cap (C \times B)$, which is a relation from C to B , is called the restriction of f to C , and is sometimes denoted $f \upharpoonright C$. In other words,

$$f \upharpoonright C = f \cap (C \times B).$$

- a) Prove that $f \upharpoonright C$ is a function from C to B and that for all $c \in C$, we have $(f \upharpoonright C)(c) = f(c)$.
- b) Suppose $g : C \rightarrow B$. Prove that $g = f \upharpoonright C$ if and only if $g \subseteq f$.

Solution—

- a) First of all, f is a relation from C to B , so $f \upharpoonright C$ is a relation from C to B . Now we need to show that for each $c \in C$, there is exactly one $b \in B$ such that $(c, b) \in f \upharpoonright C$. Since f is a function from A to B , for each $c \in C \subseteq A$, there is exactly one $b \in B$ such that $(c, b) \in f$. Therefore, $(c, b) \in f \upharpoonright C$. This shows that $f \upharpoonright C$ is a function from C to B . Moreover, for all $c \in C$, we have $(f \upharpoonright C)(c) = f(c)$ by the definition of restriction.
- b) $(\Rightarrow)$ Suppose $g = f \upharpoonright C$. Then for all $c \in C$, we have $g(c) = (f \upharpoonright C)(c) = f(c)$. Therefore, for all $(c, b) \in g$, we have $(c, b) \in f$, which means $g \subseteq f$.
- $(\Leftarrow)$ Suppose $g \subseteq f$. Then for all $(c, b) \in g$, we have $(c, b) \in f$. Since g is a function from C to B , for each $c \in C$, there is exactly one $b \in B$ such that $(c, b) \in g$. Therefore, for each $c \in C$, we have $(c, g(c)) \in f$. This means that $(c, g(c)) \in f \upharpoonright C$. Therefore, for all $c \in C$, we have $g(c) = (f \upharpoonright C)(c)$, which means that $g = f \upharpoonright C$.

Question 11— Suppose $f : A \rightarrow B$ and S is a relation on B . Define a relation R on A as follows:

$$R = \{(x, y) \in A \times A \mid (f(x), f(y)) \in S\}.$$

- a) Prove that if S is reflexive on B , then R is reflexive on A .
- b) Prove that if S is symmetric on B , then R is symmetric on A .
- c) Prove that if S is transitive on B , then R is transitive on A .

Solution—

- a) Suppose S is reflexive on B . Then for all $b \in B$, we have $(b, b) \in S$. Consider any $a \in A$. Since $f(a) \in B$, we have $(f(a), f(a)) \in S$. Therefore, $(a, a) \in R$. Since this is true for all $a \in A$, we conclude that R is reflexive on A .
- b) Suppose S is symmetric on B . Then for all $(b_1, b_2) \in S$, we have $(b_2, b_1) \in S$. Consider any $(a_1, a_2) \in R$. Then $(f(a_1), f(a_2)) \in S$. Since S is symmetric, we have $(f(a_2), f(a_1)) \in S$. Therefore, $(a_2, a_1) \in R$. Since this is true for all $(a_1, a_2) \in R$, we conclude that R is symmetric on A .
- c) Suppose S is transitive on B . Then for all $(b_1, b_2), (b_2, b_3) \in S$, we have $(b_1, b_3) \in S$. Consider any $(a_1, a_2), (a_2, a_3) \in R$. Then $(f(a_1), f(a_2)) \in S$ and $(f(a_2), f(a_3)) \in S$. Since S is transitive, we have $(f(a_1), f(a_3)) \in S$. Therefore, $(a_1, a_3) \in R$. Since this is true for all $(a_1, a_2), (a_2, a_3) \in R$, we conclude that R is transitive on A .

Question 14— Suppose A is non-empty and $f : A \rightarrow A$.

- a) Suppose there is some $a \in A$ such that $\forall x \in A (f(x) = a)$. (In this case, f is called a constant function.) Prove that for all $g : A \rightarrow A$, $f \circ g = f$.
- b) Suppose that for all $g : A \rightarrow A$, $f \circ g = f$. Prove that f is a constant function. (Hint: What happens if g is a constant function?)

Solution—

- a) Let $g : A \rightarrow A$ be any function. Then for all $x \in A$, we have $(f \circ g)(x) = f(g(x))$. Since f is a constant function, we have $f(g(x)) = a$ for all $x \in A$. Therefore, $(f \circ g)(x) = a = f(x)$ for all $x \in A$. This shows that $f \circ g = f$.
- b) Suppose that for all $g : A \rightarrow A$, $f \circ g = f$. Let $g : A \rightarrow A$ be a constant function with value $a \in A$. Then for all $x \in A$, we have $g(x) = a$. Therefore, $(f \circ g)(x) = f(g(x)) = f(a)$. Since $f \circ g = f$, we have $(f \circ g)(x) = f(x)$ for all $x \in A$. This means that $f(a) = f(x)$ for all $x \in A$. Hence, f is a constant function.

Suggested exercise: 16

5.2 One-to-one and Onto

A function $f : A \rightarrow B$ is called one-to-one (or injective) if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. In other words, no two different elements in the domain map to the same element in the codomain.

A function $f : A \rightarrow B$ is called onto (or surjective) if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. In other words, every element in the codomain is mapped to by at least one element in the domain.

A function that is both one-to-one and onto is called a bijection.

Question 8— This question mixes Question 8, 9 and 10 from the textbook.

Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$.

- a) Prove that if $g \circ f$ is onto, then g is onto.
- b) Prove that if $g \circ f$ is one-to-one, then f is one-to-one.
- c) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.
- d) Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.
- e) Suppose $D \subset A$ and consider the restriction $f \upharpoonright D : D \rightarrow B$. Prove that if f is one-to-one, then $f \upharpoonright D$ is one-to-one.
- f) Prove that if $f \upharpoonright D$ is onto, then f is onto.
- g) Give examples to show that the converses of the last two items about $f \upharpoonright D$ are false.

- Solution—**
- a) Suppose $(a, b) \in g \circ f$. If $g \circ f$ is onto, then for every $c \in C$ there is $a \in A$ with $g(f(a)) = c$. Setting $b = f(a)$ gives $g(b) = c$, so g is onto.
 - b) Suppose $f(a_1) = f(a_2)$. Applying g gives $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$.
 - c) Since g is not one-to-one, choose $b_1 \neq b_2$ with $g(b_1) = g(b_2)$. Since f is onto, choose a_1, a_2 with $f(a_i) = b_i$. Then $a_1 \neq a_2$ and $(g \circ f)(a_1) = (g \circ f)(a_2)$, so $g \circ f$ is not one-to-one.
 - d) Since f is not onto, choose $b_0 \in B \setminus f(A)$. Since g is one-to-one, $g(b_0) \notin g(f(A))$, which is the range of $g \circ f$. Thus $g \circ f$ is not onto.
 - e) If f is one-to-one and $c_1, c_2 \in D$ satisfy $f(c_1) = f(c_2)$, then $c_1 = c_2$. Hence $f \upharpoonright D$ is one-to-one.

- f) If $f \upharpoonright D$ is onto B , then every $b \in B$ equals $f(d)$ for some $d \in D \subseteq A$. Therefore f is onto.
- g) The converse of (5) is false: let $A = \{1, 2\}$, $D = \{1\}$, $B = \{1\}$, and $f(1) = f(2) = 1$. Then $f \upharpoonright D$ is one-to-one, but f is not one-to-one. The converse of (6) is false: let $A = \{1, 2\}$, $D = \{1\}$, $B = \{1, 2\}$, and $f(1) = 1, f(2) = 2$. Then f is onto, but $f \upharpoonright D$ is not onto B .

6 Mathematical induction

In order to prove a statement of the form “for all $n \in \mathbb{N}$, $P(n)$ is true”, we can use the principle of mathematical induction. The principle of mathematical induction states that if we can prove the following two statements:

- a) $P(0)$ is true (base case)
- b) For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true (inductive step)

then we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Some variants of the principle of mathematical induction include starting the base case at $n = \tilde{n}$ instead of $n = 0$ in order to prove that $P(n)$ is true for all $n \geq \tilde{n}$, and strong induction, where we assume that $P(k)$ is true for all $k \leq n$ in order to prove that $P(n+1)$ is true. Note that simple induction and strong induction are equivalent, in the sense that if a statement can be proved using one method, it can also be proved using the other method.

6.1 Proof by Mathematical Induction

Question 1— Prove that for all $n \in \mathbb{N}$, $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Solution— (Base case) For $n = 0$, we have $0 = \frac{0(0+1)}{2} = 0$. Therefore, the base case holds.

(Inductive step) Suppose that for some $k \in \mathbb{N}$, we have $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. We want to show that $0 + 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$.

$$\begin{aligned} 0 + 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Therefore, the inductive step holds.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Question 2— Prove that for all $n \in \mathbb{N}$, $0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution— (Base case) For $n = 0$, we have $0^2 = \frac{0(0+1)(2 \cdot 0 + 1)}{6} = 0$. Therefore, the base case holds.

(Inductive step) Suppose that for some $k \in \mathbb{N}$, we have $0^2 + 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$. We

want to show that $0^2 + 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

$$\begin{aligned}
 0^2 + 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
 &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
 &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
 &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
 \end{aligned}$$

Therefore, the inductive step holds.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Question 3— Prove that for all $n \in \mathbb{N}$, $0^3 + 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Solution— (Base case) For $n = 0$, we have $0^3 = \left(\frac{0(0+1)}{2}\right)^2 = 0$. Therefore, the base case holds.

(Inductive step) Suppose that for some $k \in \mathbb{N}$, we have $0^3 + 1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. We want to show that $0^3 + 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$.

$$\begin{aligned}
 0^3 + 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
 &= (k+1)^2 \left(\frac{k^2}{4} + (k+1)\right) \\
 &= (k+1)^2 \left(\frac{k^2 + 4(k+1)}{4}\right) \\
 &= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4}\right) \\
 &= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) \\
 &= \left(\frac{(k+1)(k+2)}{2}\right)^2
 \end{aligned}$$

Therefore, the inductive step holds.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Question 4— Find a formula for $1 + 3 + 5 + \cdots + (2n-1)$, for $n \geq 1$, and prove that your formula is correct. (Hint: First try some particular values of n and look for a pattern.)

Solution— We start by computing the sum for some small values of n :

- For $n = 1$, we have 1.
- For $n = 2$, we have $1 + 3 = 4$.
- For $n = 3$, we have $1 + 3 + 5 = 9$.
- For $n = 4$, we have $1 + 3 + 5 + 7 = 16$.

It seems like the sum of the first n odd numbers is n^2 . We will prove this by induction.

(Base case) The result holds for $n = 1$. Actually, we showed it for $n \leq 4$.

(Induction step) Assume that the result holds for some $k \in \mathbb{N}$. That is, assume that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

We want to show that the result also holds for $k + 1$. That is, we want to show that

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

Adding $(2k + 1)$ to both sides of the inductive hypothesis, we get

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1).$$

Simplifying the right-hand side, we get

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.$$

Therefore, the inductive step holds.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Question 5— Prove that for all $n \in \mathbb{N}$, $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$.

Solution— We will prove this by induction.

(Base case) The result holds for $n = 0$. Indeed, the left-hand side is $0 \cdot 1 = 0$, and the right-hand side is $\frac{0(0+1)(0+2)}{3} = 0$.

(Induction step) Assume that the result holds for some $k \in \mathbb{N}$. That is, assume that

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1) = \frac{k(k + 1)(k + 2)}{3}.$$

We want to show that the result also holds for $k + 1$. That is, we want to show that

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1) + (k + 1)(k + 2) = \frac{(k + 1)(k + 2)(k + 3)}{3}.$$

Adding $(k + 1)(k + 2)$ to both sides of the inductive hypothesis, we get

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1) + (k + 1)(k + 2) = \frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2).$$

Factoring out $(k+1)(k+2)$ from the right-hand side, we get

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = (k+1)(k+2) \left(\frac{k}{3} + 1 \right).$$

Simplifying the expression inside the parentheses, we get

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = (k+1)(k+2) \left(\frac{k+3}{3} \right).$$

Therefore, we have

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

This completes the proof.